

# THE BASS–SERRE TREES OF AMALGAMATED FREE PRODUCT $C^*$ -ALGEBRAS

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**ABSTRACT.** For any reduced amalgamated free product  $C^*$ -algebra  $(A, E) = (A_1, E_1) *_D (A_2, E_2)$ , we introduce a canonical ambient  $C^*$ -algebra  $\Delta \mathbf{T}(A, E)$  of  $A$  in such a way that if each  $(A_k, E_k)$  comes from discrete groups  $\Lambda \leq \Gamma_k$ , then the inclusion  $A \subset \Delta \mathbf{T}(A, E)$  is identical to  $C_{\text{red}}^*(\Gamma) \subset C(\Delta \mathbf{T}) \rtimes_{\text{red}} \Gamma$  with the canonical action of  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  on the Bowditch compactification  $\Delta \mathbf{T}$  of the Bass–Serre tree  $\mathbf{T}$  associated with  $\Gamma$ . We then identify  $\Delta \mathbf{T}(A, E)$  with a Cuntz–Pimsner algebra in a very explicit way. This identification result provides new conceptual, and simpler proofs of several known theorems on approximation properties and  $KK$ -theory for reduced amalgamated free product  $C^*$ -algebras.

## 1. INTRODUCTION

Any amalgamated free product group  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  is known to act on the associated Bass–Serre tree. This type of action is one of two fundamental examples of group actions on trees in Bass–Serre theory [Se] (the other arising from HNN extensions), and also plays an important role in  $K$ -theory of those groups. In fact, Julg and Valette [JV] gave a geometric construction of  $1_\Gamma \in KK_\Gamma(\mathbb{C}, \mathbb{C})$  using the action on the tree, which turned out to be the gamma element in the context of the Baum–Connes conjecture ([Va]). Also, one can compute the  $KK$ -theory of  $\Gamma$  from those of  $\Gamma_1, \Gamma_2$  and  $\Lambda$  by Pimsner’s result [Pi1].

Julg and Valette’s geometric construction was applied to amalgamated free products of quantum discrete groups in [Ve], and then to the general  $C^*$ -algebras in [Ha, FG1]. In these works, natural analogues of the Bass–Serre trees for reduced amalgamated free products of  $C^*$ -algebras appear and play a key role. This consideration is further generalized to a framework of graph of  $C^*$ -algebras in [FF], and Fima and Germain generalized Pimsner’s result [Pi1] to this framework [FG2].

In the present paper, based on these ideas, we study reduced amalgamated free products from a more geometric viewpoint via associated Bass–Serre trees. To explain our result, let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product of discrete groups acting on the Bass–Serre tree  $\mathbf{T}$ . Then  $\Gamma$  admits a natural action on the Bowditch compactification  $\Delta \mathbf{T}$ , which is the set of equivalence classes of geodesic paths in  $\mathbf{T}$  with a suitable topology [Bo]. This action can be thought of as an analogue of boundary actions of word hyperbolic groups, and its boundary amenability was studied by Ozawa [Oz2]. The main result of this paper is to generalize the inclusion  $C_{\text{red}}^*(\Gamma) \subset C(\Delta \mathbf{T}) \rtimes_{\text{red}} \Gamma$  of  $C^*$ -algebras to general reduced amalgamated free products. Our key observations are the following:

- (i) The reduced crossed product  $C(\Delta \mathbf{T}) \rtimes_{\text{red}} \Gamma$  is generated by a copy of  $\Gamma$  and two projections  $P_1$  and  $P_2$  with  $P_1 + P_2 = 1$  on a Hilbert  $C^*$ -module over  $C_{\text{red}}^*(\Gamma)$ .
- (ii) The definition of projections  $P_1$  and  $P_2$  needs only the reduced amalgamated free product structure of  $C_{\text{red}}^*(\Gamma)$ .

In § 3 we verify these observations and construct a canonical  $C^*$ -algebra  $\Delta \mathbf{T}(A, E)$  for a general reduced amalgamated free product  $(A, E) = (A_1, E_1) *_D (A_2, E_2)$ . Although our  $C^*$ -algebra is not a crossed product  $C^*$ -algebra, we have the next identification result.

**Theorem A.** *The  $C^*$ -algebra  $\Delta\mathbf{T}(A, E)$  is isomorphic to the Cuntz–Pimsner algebra  $\mathcal{O}(\mathfrak{X})$  of a  $C^*$ -correspondence  $\mathfrak{X}$  over  $B_1 \oplus B_2$ , where  $B_1$  and  $B_2$  are semisplit extensions of  $A_1$  and  $A_2$  by the compacts on some Hilbert  $C^*$ -modules over  $D$ , respectively. Moreover, the Toeplitz extension of  $\mathcal{O}(\mathfrak{X})$  is semisplit.*

This theorem is proved in § 4. One may think that the statement of Theorem A is too technical, but it characterizes when  $\Delta\mathbf{T}(A, E)$  is nuclear or exact (Corollary 4.3.4) and immediately gives the next known fact due to Dykema [Dy] as a corollary.

**Corollary B** (Dykema). *Reduced amalgamated free products of exact  $C^*$ -algebras are exact.*

Theorem A is in the same spirit of [Spi, Ok] for boundary actions of amalgamated free products of finite groups. In fact, we show in § 5 that  $\Delta\mathbf{T}(A, E)$  contains a canonical ideal corresponding to  $c_0(\mathbf{V}) \cap C(\Delta\mathbf{T}) \rtimes_{\text{red}} \Gamma$ , where  $\mathbf{V}$  is the vertex set of the tree  $\mathbf{T}$ , and the quotient of  $\Delta\mathbf{T}(A, E)$  by that ideal is again isomorphic to some Cuntz–Pimsner algebra. This generalizes Okayasu’s result that  $C(\partial\mathbf{T}) \rtimes \Gamma$  is isomorphic to some Cuntz–Pimsner algebra when  $\Gamma_1$  and  $\Gamma_2$  are finite groups.

In § 6, using the six-term exact sequences of  $KK$ -groups induced from the semisplit Toeplitz extension, we give a new proof of Fima and Germain’s result [FG1]:

**Theorem C** (Fima–Germain). *Let  $(A, E) = (A_1, E_1) *_D (A_2, E_2)$  be a reduced amalgamated free product of unital separable  $C^*$ -algebras and  $i_k: D \hookrightarrow A_k$  and  $j_k: A_k \hookrightarrow A$  be inclusion maps for  $k = 1, 2$ . Then, there are two cyclic exact sequences for any separable  $C^*$ -algebra  $P$ :*

$$\begin{array}{ccccc} KK(P, D) & \xrightarrow{(i_{1*}, i_{2*})} & KK(P, A_1) \oplus KK(P, A_2) & \xrightarrow{j_{1*} - j_{2*}} & KK(P, A) \\ \uparrow & & & & \downarrow \\ KK^1(P, A) & \xleftarrow{j_{1*} - j_{2*}} & KK^1(P, A_1) \oplus KK^1(P, A_2) & \xleftarrow{(i_{1*}, i_{2*})} & KK^1(P, D) \end{array}$$

and

$$\begin{array}{ccccc} KK(D, P) & \xleftarrow{i_1^* - i_2^*} & KK(A_1, P) \oplus KK(A_2, P) & \xleftarrow{(j_1^*, j_2^*)} & KK(A, P) \\ \downarrow & & & & \uparrow \\ KK^1(A, P) & \xrightarrow{(j_1^*, j_2^*)} & KK^1(A_1, P) \oplus KK^1(A_2, P) & \xrightarrow{i_1^* - i_2^*} & KK^1(D, P). \end{array}$$

Although our proof is indirect compared to the original proof, it needs only Puppe exact sequences for mapping cones and six-term exact sequences induced from semisplit exact sequences [Sk, CS] with elementary diagram chasing. Finally, we show that following:

**Theorem D.** *Let  $\phi: A \hookrightarrow \Delta\mathbf{T}(A, E)$  be the inclusion map. Then, there exists a non-unital injective  $*$ -homomorphism  $\rho_1: D \hookrightarrow \Delta\mathbf{T}(A, E)$  such that  $\phi \oplus \rho_1 \in KK(A \oplus D, \Delta\mathbf{T}(A, E))$  is a  $KK$ -equivalence.*

Note that this theorem implies that the  $KK$ -class of  $\Delta\mathbf{T}(A, E)$  is independent of the choice of conditional expectations  $E_1$  and  $E_2$  by the  $KK$ -equivalence between  $A$  and the corresponding full amalgamated free product by [Ha, FG1].

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## 2. PRELIMINARIES

**2.1. Notations on Hilbert C\*-modules.** We refer to Lance's book [La] for the theory of Hilbert C\*-modules. For a Hilbert C\*-module  $X$ , we denote by  $\mathbb{L}(X)$  the C\*-algebra of all adjointable operators on  $X$ . The “rank one operator” associated with  $\xi, \eta \in X$  is denoted by  $\theta_{\xi, \eta}$  and the elements of  $\mathbb{K}(X) := \overline{\text{span}}\{\theta_{\xi, \eta} \mid \xi, \eta \in X\}$  are called compact operators.

Let  $X$  and  $Y$  be Hilbert C\*-modules over  $A$  and  $B$ , respectively. We denote by  $X \boxplus Y$  the direct product of  $X$  and  $Y$  equipped with the  $A \oplus B$ -valued inner product  $\langle \xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \oplus \langle \eta_1, \eta_2 \rangle$  for  $\xi_k \in X$  and  $\eta_k \in Y$ . Note that  $\mathbb{L}(X \boxplus Y) \cong \mathbb{L}(X) \oplus \mathbb{L}(Y)$ . If  $(Y, \phi_Y)$  is an  $A$ - $B$  C\*-correspondence, we denote the interior tensor product of  $X$  and  $(Y, \phi_Y)$  by  $X \otimes_A Y$ . For each  $x \in \mathbb{L}(X)$ , we denote by  $x \otimes 1$  the image of  $x$  under the natural map  $\mathbb{L}(X) \rightarrow \mathbb{L}(X \otimes_B Y)$ .

**2.2. Reduced amalgamated free products.** We fix notations on reduced amalgamated free products which will be used throughout the paper. Let  $\{(D \subset A_k, E_k)\}_{k \in \mathcal{I}}$  be a family of unital inclusions of C\*-algebras with nondegenerate conditional expectations and  $(A, E) = \bigstar_D (A_k, E_k)$  be the reduced amalgamated free product (see [Vo]). We denote by  $(X_k, \phi_{X_k}, \xi_k)$  and  $(X, \phi_X, \xi_0)$  the GNS representations associated with  $E_k$  and  $E$ , respectively. For each  $m \in \mathbb{N}$ , we set  $\mathcal{I}_m := \{\iota : \{1, \dots, m\} \rightarrow \mathcal{I} \mid \iota(j) \neq \iota(j+1) \text{ for } 1 \leq j \leq m-1\}$ . We set  $X_k^\circ = X_k \ominus \xi_k D$  and  $a^\circ = a - E_k(a)$  for  $a \in A_k$ . Then,  $X$  is identified with the free product Hilbert C\*-module

$$\xi_0 D \oplus \bigoplus_{m \geq 1} \bigoplus_{\iota \in \mathcal{I}_m} X_{\iota(1)}^\circ \otimes_D \cdots \otimes_D X_{\iota(m)}^\circ.$$

For each  $k \in \mathcal{I}$  we denote by  $P_{(\ell, k)}$  and  $P_{(r, k)}$  the projections onto the following submodules, respectively:

$$\begin{aligned} X(\ell, k) &:= \xi_0 D \oplus \bigoplus_{m \geq 1} \bigoplus_{\substack{\iota \in \mathcal{I}_m \\ \iota(1) \neq k}} X_{\iota(1)}^\circ \otimes_D \cdots \otimes_D X_{\iota(m)}^\circ, \\ X(r, k) &:= \xi_0 D \oplus \bigoplus_{m \geq 1} \bigoplus_{\substack{\iota \in \mathcal{I}_m \\ \iota(m) \neq k}} X_{\iota(1)}^\circ \otimes_D \cdots \otimes_D X_{\iota(m)}^\circ. \end{aligned}$$

The compression by the projection onto  $\xi_0 D \oplus X_k^\circ \cong X_k$  defines the conditional expectation  $E_{A_k} : A \rightarrow A_k$  such that  $E = E_k \circ E_{A_k}$ . We denote by the GNS representation of  $E_{A_k}$  by  $(Y_k, \phi_{Y_k}, \eta_k)$ . Then, there exists a unitary  $S_k : X(r, k) \otimes_D A_k \rightarrow Y_k$  such that  $S_k a_1 \cdots a_m \xi_0 \otimes b = a_1 \cdots a_m \eta_k b$  for all  $b \in A_k$  and  $m \in \mathbb{N}$ ,  $\iota \in \mathcal{I}_m$  with  $\iota(m) \neq k$  and  $a_i \in A_{\iota(i)}^\circ$  for  $1 \leq i \leq m$  (see [Ha, Lemma 3.1.1]). We use the following  $A$ - $\prod_{k \in \mathcal{I}} A_k$  and  $A$ - $A$  C\*-correspondences

$$(Y, \phi_Y) = \bigsqcup_{k \in \mathcal{I}} (Y_k, \phi_{Y_k}), \quad (Z, \phi_Z) = \bigoplus_{k \in \mathcal{I}} (Y_k \otimes_{A_k} A, \phi_{Y_k} \otimes 1), \quad (1)$$

and may identify  $A$  with  $\phi_Y(A)$ . Note that there exists an injective \*-homomorphism  $\tilde{\phi}_Z$  from  $\mathbb{L}(Y)$  to  $\mathbb{L}(Z)$  such that  $\phi_Z = \tilde{\phi}_Z \circ \phi_Y$ .

**2.3. Pimsner algebras.** We correct notations and terminologies on Pimsner algebras following [Ka]. Let  $(X, \phi_X)$  be a C\*-correspondence over a C\*-algebra  $A$ . We do *not* assume that  $\phi_X$  is injective. Recall that a *representation* of  $X$  on a C\*-algebra  $B$  is a pair  $(\pi, t)$  such that  $\pi : A \rightarrow B$  is a \*-homomorphism and  $t : X \rightarrow B$  is a linear map satisfying  $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle)$  and  $\pi(a) t(\xi) \pi(b) = t(\phi_X(a) \xi b)$  for  $\xi, \eta \in X$  and  $a, b \in A$ . For each  $n \in \mathbb{N}$ , we denote by  $X^{\otimes n}$  the interior tensor product  $X \otimes_A X \otimes_A \cdots \otimes_A X$  of  $n$  copies of  $X$ , and set  $X^{\otimes 0} = A$  and  $t^0 = \pi$ . Then, there is a linear map  $t^n : X^n \rightarrow B$  such that  $t^n(\xi_1 \otimes \cdots \otimes \xi_n) = t(\xi_1) \cdots t(\xi_n)$  for  $\xi_1, \dots, \xi_n \in X$ . The C\*-algebra  $C^*(\pi, t)$  generated by  $\pi(A)$  and  $t(X)$  is the closed linear span of

$$\{t^m(\xi) t^n(\eta)^* \mid m, n \in \mathbb{N} \cup \{0\}, \xi \in X^{\otimes m}, \eta \in X^{\otimes n}\}.$$

When  $(\pi, t)$  is the universal representation (which is unique up to isomorphism),  $C^*(\pi, t)$  is called the *Toeplitz–Pimsner algebra* of  $X$ , and denoted by  $\mathcal{T}(X)$ . Let  $\mathcal{F}(X) = \bigoplus_{n \geq 0} X^{\otimes n}$  be the full Fock space and  $(\varphi_\infty, \tau)$  be the Fock representation of  $X$  on  $\mathbb{L}(\mathcal{F}(X))$  defined by  $\varphi_\infty(a) = a \oplus \phi_X(a) \oplus \bigoplus_{n \geq 2} \phi_X(a) \otimes 1$  and  $\tau(\xi)a = \xi a$  and  $\tau(\xi)\eta = \xi \otimes \eta$  for  $a \in A$ ,  $\xi \in X$  and  $\eta \in X^{\otimes n}$  with  $n \geq 1$ . Then,  $(\varphi_\infty, \tau)$  is universal, so we have  $\mathcal{T}(X) \cong C^*(\varphi_\infty, \tau)$ . The compression map by the projection onto  $A \subset \mathcal{F}(X)$  defines a nondegenerate conditional expectation  $E_X: \mathcal{T}(X) \rightarrow A$ .

Any representation  $(\pi, t)$  induces a  $*$ -homomorphism  $\psi_t: \mathbb{K}(X) \rightarrow B$  such that  $\psi_t(\theta_{\xi, \eta}) = t(\xi)t(\eta)^*$ . We define the ideal  $J_X$  of  $A$  by

$$\phi_X^{-1}(\mathbb{K}(X)) \cap (\ker \phi_X)^\perp = \{a \in \phi_X^{-1}(\mathbb{K}(X)) \mid ax = 0 \text{ for } x \in \ker \phi_X\}.$$

We say that  $(\pi, t)$  is *covariant* if  $\pi = \psi_t \circ \phi_X$  holds on  $J_X$ . When  $(\pi, t)$  is a universal covariant representation,  $C^*(\pi, t)$  is called the *Cuntz–Pimsner algebra* and denoted by  $\mathcal{O}(X)$ . Note that the definition of  $\mathcal{O}(X)$  is different from Pimsner’s original one in [Pi2]. By the universality, there exists a quotient map  $\mathcal{T}(X) \rightarrow \mathcal{O}(X)$ , and the kernel is isomorphic to  $\mathbb{K}(\mathcal{F}(X)J_X)$  via  $\mathcal{T}(X) \cong C^*(\varphi_\infty, \tau)$ .

### 3. COMPACTIFICATIONS OF BASS–SERRE TREES

**3.1. Group case.** Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product of discrete groups and set  $\mathcal{I} = \{1, 2\}$ . Recall that the *Bass–Serre tree* of  $\Gamma$  is the graph  $\mathbf{T} = (\mathbf{V}, \mathbf{E})$  of which the vertex set is  $\mathbf{V} = \Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2$  and the edge set is  $\mathbf{E} = \Gamma/\Lambda$ , such that the edge  $g\Lambda$  relates  $g\Gamma_1$  and  $g\Gamma_2$  (see [Se]). We denote by  $\Delta\mathbf{T} := \mathbf{V} \sqcup \partial\mathbf{T}$  the Bowditch compactification of  $\mathbf{T}$  as a uniformly fine hyperbolic graph (see [Bo] or [Oz2, §2]). If we identify each element in  $\mathbf{V}$  with the finite geodesic path in  $\mathbf{T}$  connecting it with the origin  $e\Gamma_1 \in \mathbf{V}$  (the coset of  $\Gamma_1$  with respect to the neutral element), then elements in  $\partial\mathbf{T}$  are identified with one-sided infinite geodesic paths from  $e\Gamma_1$ . For each  $x, y \in \Delta\mathbf{T}$  we denote by  $[x, y] \subset \Delta\mathbf{T}$  the unique geodesic path connecting  $x$  and  $y$ , and for each finite subset  $F$  of  $V$ , we set

$$M(x, F) := \{y \in \Delta\mathbf{T} \mid [x, y] \cap F = \emptyset\}.$$

Then,  $M(x, F)$  is clopen in  $\Delta\mathbf{T}$  and the family  $\{M(x, F) \mid x \in \Delta\mathbf{T}, F \subset \mathbf{V} \text{ finite}\}$  forms an open base for  $\Delta\mathbf{T}$ . We have natural inclusions  $C(\Delta\mathbf{T}) \subset \ell^\infty(\mathbf{V}) \subset \mathbb{B}(\ell^2(\mathbf{V}))$  and the action  $\alpha: \Gamma \curvearrowright C(\Delta\mathbf{T})$  is implemented by the unitary representation  $\pi: \Gamma \curvearrowright \ell^2(\mathbf{V})$  induced from the action  $\Gamma \curvearrowright \mathbf{T}$ . Note that  $(\ell^2(\mathbf{V}), \pi)$  is unitarily equivalent to the direct sum of quasiregular representations  $(\ell^2(\Gamma/\Gamma_k), \lambda_{\Gamma/\Gamma_k})$ ,  $k = 1, 2$ .

Let  $A, A_k$  and  $D$  be the reduced group  $C^*$ -algebras of  $\Gamma, \Gamma_k$  and  $\Lambda$ , respectively, and  $E_k: A_k \rightarrow D$  and  $E_{A_k}: A \rightarrow A_k$  be the canonical conditional expectations. Then, one has  $(A, E) \cong (A_1, E_1) *_D (A_2, E_2)$ . We use the notations in §§ 2.2 and identify  $A$  with  $\phi_Y(A) \subset \mathbb{L}(Y)$ . Define the projection  $P_k \in \mathbb{L}(Y)$  for  $k \in \mathcal{I}$  by

$$P_k^\circ := \sum_{j \in \mathcal{I}} S_j(P_{\ell, k}^\perp \wedge P_{(r, k)} \otimes 1)S_j^*, \quad P_k := e_{A_k} + P_k^\circ, \quad (2)$$

where  $e_{A_k} \in \mathbb{L}(Y_k) \subset \mathbb{L}(Y)$  is the Jones projection of  $E_{A_k}$ .

**Proposition 3.1.1.** *The reduced crossed product  $C(\Delta\mathbf{T}) \rtimes_{\text{red}} \Gamma$  is isomorphic to  $C^*(A, P_1, P_2)$ .*

*Proof.* Let  $P_{\Gamma_1}$  and  $P_{\Gamma_2}$  be the projections in  $C(\Delta\mathbf{T})$  obtained by cutting off the edge  $e\Lambda$  such that  $P_{\Gamma_k} \delta_{e\Gamma_k} = \delta_{e\Gamma_k}$  for  $k = 1, 2$ . Then, the range of  $P_{\Gamma_k}$  is generated by all the vectors  $\delta_{g\Gamma_j}$  such that  $j \in \{1, 2\}$  and  $g$  is a reduced word beginning with some element in  $\Gamma_k \setminus \Lambda$ . Then, the  $\Gamma$ -orbits of  $P_{\Gamma_1}$  and  $P_{\Gamma_2}$  separate the points of  $\Delta\mathbf{T}$ . Thus, it follows from the Stone–Weierstrass theorem that

$$C(\Delta\mathbf{T}) \rtimes_{\text{red}} \Gamma \cong C^*(\{P_{\Gamma_k} \otimes 1\}_{k=1,2} \cup \{\pi(g) \otimes \lambda(g)\}_{g \in \Gamma}) \subset \mathbb{L}(\ell^2(\mathbf{V}) \otimes A).$$

For each  $k \in \{1, 2\}$ , we define the unitary  $U_k: \ell^2(\Gamma/\Gamma_k) \otimes A \rightarrow Y_k \otimes_{A_k} A$  by  $U_k(\delta_{g\Gamma_k} \otimes a) = \lambda(g)\eta_k \otimes \lambda(g)^*a$  for  $g \in \Gamma$  and  $a \in A$ . Then, one can easily check that  $U_k(\lambda_{\Gamma/\Gamma_k}(g) \otimes \lambda(g)) = (\phi_{Y_k}(\lambda(g)) \otimes 1)U_k$  for  $g \in \Gamma$ . Let  $(Z, \phi_Z)$  and  $\tilde{\phi}_Z: \mathbb{L}(Y) \rightarrow \mathbb{L}(Z)$  be as in Eq.(1). Letting  $U := U_1 \oplus U_2: \ell^2(V) \otimes A \rightarrow Z$  we obtain

$$U(\pi(g) \otimes \lambda(g))U^* = \phi_Z(\lambda(g)) \quad \text{for } g \in \Gamma, \quad \tilde{\phi}_Z(P_k) = UP_{\Gamma_k}U^* \quad \text{for } k = 1, 2.$$

Since  $\tilde{\phi}_Z$  is injective,  $C(\Delta\mathbf{T}) \rtimes_{\text{red}} \Gamma$  is isomorphic to  $C^*(A, P_1, P_2)$ .  $\square$

By a similar argument to the above proof, one can show that the  $C^*$ -correspondence  $\ell^2(\mathbf{E}) \otimes A$  equipped with the left action  $\lambda(g) \mapsto \lambda_{\Gamma/\Lambda}(g) \otimes \lambda(g)$  is unitarily equivalent to  $(X \otimes_D A, \phi_X \otimes 1)$ . Therefore, the pair  $(Z, \phi_Z)$  and  $(X \otimes_D A, \phi_X \otimes 1)$  can be regarded as the Bass-Serre tree of  $(A, E)$  (c.f. [Ve, FF, FG1, FG2]).

**3.2. General case.** Let  $(A, E) = \bigstar_D(A_k, E_k)$  a reduced amalgamated free product. We assume that  $A$  is a subalgebra of  $\mathbb{L}(Y)$ . For each  $k \in \mathcal{I}$ , we define the projection  $P_k$  in  $\mathbb{L}(Y)$  by Eq. (2).

**Definition 3.2.1.** We define  $\Delta\mathbf{T}(A, E)$  by the  $C^*$ -algebra generated by  $A$  and  $\{P_k\}_{k \in \mathcal{I}}$  inside  $\mathbb{L}(Y)$ .

**Remark 3.2.2.** Let  $\Gamma = \bigstar_{\Lambda} \Gamma_k$  be the amalgamated free product of  $\{\Lambda \leq \Gamma_k\}_{k \in \mathcal{I}}$ . When  $|\mathcal{I}| \geq 3$ , the Bass-Serre tree of  $\Gamma$  is given by the following way: Let 0 be an element which is not in  $\mathcal{I}$  and set  $\mathcal{J} = \mathcal{I} \cup \{0\}$  and  $\Gamma_0 := \Lambda$ . Then, the associated Bass-Serre tree  $\mathbf{T}^\sim = (\mathbf{V}^\sim, \mathbf{E}^\sim)$  is given by  $\mathbf{V}^\sim = \bigsqcup_{k \in \mathcal{J}} \Gamma/\Gamma_k$  and  $\mathbf{E}^\sim = \bigsqcup_{k \in \mathcal{I}} \Gamma/\Lambda_k$  with  $\Lambda_k := \Lambda$  for  $k \in \mathcal{I}$ . Here  $g\Lambda_k$  relates  $g\Gamma_k$  and  $g\Lambda$  for  $g \in \Gamma$ .

Let  $(A, E) = \bigstar_D(A_k, E_k)$  be the corresponding reduced amalgamated free product. Note that the  $C^*$ -algebra  $\Delta\mathbf{T}(A, E)$  constructed above is different from  $C(\mathbf{T}^\sim) \rtimes_{\text{red}} \Gamma$ . Let  $P_{\Gamma_k}$  be the projection in  $\ell^\infty(\mathbf{V})$  corresponding to the clopen set  $\Delta\mathbf{T} \setminus M(\Lambda, \{\Gamma_k\})$  for  $k \in \mathcal{I}$ . Then, one can show that  $C(\Delta\mathbf{T}) \rtimes_{\text{red}} \Gamma$  is isomorphic to the  $C^*$ -algebra  $\Delta\mathbf{T}(A, E)^\sim$  generated by  $\phi_Y \oplus \phi_X(A)$  and  $P_k \oplus P_{(\ell, k)}^\perp$ ,  $k \in \mathcal{I}$  inside  $\mathbb{L}(Y \boxplus X)$ . Note that  $\Delta\mathbf{T}(A, E)$  is the image of  $\Delta\mathbf{T}(A, E)^\sim$  under the quotient map  $\mathbb{L}(Y \boxplus X) \rightarrow \mathbb{L}(Y)$ .

The next proposition follows from the definition, so we omit the proof.

**Proposition 3.2.3.** *The following hold true:*

- (i) *The projections  $P_k, k \in \mathcal{I}$  commute with  $D$ .*
- (ii) *Any element  $a \in A_k$  enjoys  $P_k^\perp a P_k^\perp = E(a)P_k^\perp$ .*
- (iii) *Any element  $a \in A_k^\circ$  enjoys  $a P_k^\perp = P_k^\circ a P_k^\perp$ .*
- (vi) *The compression  $\mathbb{L}(Y) \rightarrow \mathbb{L}(\eta_k A_k) \cong A_k$  by  $e_{A_k}$  defines a conditional expectation from  $\Delta\mathbf{T}(A, E)$  onto  $A_k$  extending  $E_{A_k}$ .*

**Remark 3.2.4.** We will use the following representations of  $\Delta\mathbf{T}(A, E)$ :

- (i) For each  $k \in \mathcal{I}$ , we consider the  $\Delta\mathbf{T}(A, E)$ - $D$   $C^*$ -correspondence  $X^{(k)} = Y_k \otimes_{A_k} X_k \cong X$  with the left action  $\sigma_k$  defined by the composition of the cut-off  $\mathbb{L}(Y) \cong \prod_{i \in \mathcal{I}} \mathbb{L}(Y_i) \rightarrow \mathbb{L}(Y_k)$  and the “ $\otimes 1$  map”  $\mathbb{L}(Y_k) \rightarrow \mathbb{L}(Y_k \otimes_{A_k} X_k)$ . We observe that for each  $k, j \in \mathcal{I}$  with  $k \neq j$  one has

$$\sigma_k|_A = \phi_X, \quad \sigma_k(P_k) = e_D + P_{(\ell, k)}^\perp, \quad \sigma_j(P_k) = P_{(\ell, k)}^\perp.$$

- (ii) Consider the unitary  $V := \sum_{k \in \mathcal{I}} S_k: Y \rightarrow \bigoplus_{k \in \mathcal{I}} X(r, k) \otimes_D A_k$ . Then,  $\Delta\mathbf{T}(A, E)$  is isomorphic to  $C^*(V^* A_k V, V P_k V^*, k \in \mathcal{I})$ . For  $k, j \in \mathcal{I}$  with  $k \neq j$  and  $a \in A_k$ , we have

$$V^* a V = \begin{cases} a \oplus (\phi_X(a) \otimes 1) & \text{on } A_k \oplus (X(r, k)^\circ \otimes_D A_k); \\ \phi_X(a) \otimes 1 & \text{on } X(r, j) \otimes_D A_j. \end{cases}$$

$$V^* P_k V = \begin{cases} (e_D \otimes 1) \oplus (P_{(\ell, k)}^\perp \wedge P_{(r, k)} \otimes 1) & \text{on } (\xi_0 D \otimes_D A_k) \oplus (X(r, k)^\circ \otimes_D A_k); \\ P_{(\ell, k)}^\perp \wedge P_{(r, j)} \otimes 1 & \text{on } X(r, j) \otimes_D A_j. \end{cases}$$

## 4. PIMSNER ALGEBRAS

**4.1. Extensions associated with conditional expectations.** Let  $D \subset A$  be a unital inclusion of  $C^*$ -algebras with conditional expectation  $E: A \rightarrow D$ . Let  $(X, \phi_X, \xi_0)$  be the GNS representation associated with  $E$  and  $e_D \in \mathbb{L}(X)$  be the Jones projection. We define the UCP map  $\Phi: A \rightarrow \mathbb{L}(X^\circ) \cong e_D^\perp \mathbb{L}(X) e_D^\perp$  by the compression, and set  $\Psi := \text{id}_A \oplus \Phi: A \rightarrow A \oplus \mathbb{L}(X^\circ)$  and  $B := C^*(\Psi(A))$ . For any  $a, b \in A^\circ$ , we have

$$\Psi(ab^*) - \Psi(a)\Psi(b^*) = 0 \oplus \theta_{\hat{a}, \hat{b}}.$$

Thus,  $0 \oplus \mathbb{K}(X^\circ) \subset B$  holds. We may identify this ideal with  $\mathbb{K}(X^\circ)$  so that  $\mathbb{K}(X^\circ) \subset B$ . The next proposition was essentially proved in [Dy], but we give its proof for the reader's convenience.

**Proposition 4.1.1.** *There is a semisplit exact sequence*

$$0 \longrightarrow \mathbb{K}(X^\circ) \longrightarrow B \longrightarrow A \longrightarrow 0 \quad (3)$$

*with the UCP cross section  $\Psi: A \rightarrow B$ . The  $C^*$ -algebra  $B$  is nuclear (resp. exact) if and only if  $A$  is nuclear (resp. exact).*

*Proof.* Let  $q: \mathbb{L}(X^\circ) \rightarrow \mathbb{L}(X^\circ)/\mathbb{K}(X^\circ)$  be the quotient map. The above computation implies that  $(\text{id} \oplus q) \circ \Psi: A \rightarrow A \oplus \mathbb{L}(X^\circ)/\mathbb{K}(X^\circ)$  is an injective  $*$ -homomorphism. The second assertion follows from the semisplit exact sequence and [Ka, Proposition B.7].  $\square$

**Definition 4.1.2.** We call the pair  $(B, \Psi)$  the *semisplit extension associated with  $(D \subset A, E)$* .

**Remark 4.1.3.** Every element in  $B$  is of the form  $\Psi(a) + K$  for some  $a \in A$  and  $K \in \mathbb{K}(X^\circ)$ . Since  $e_D$  is compact, the ideal  $I := \phi_X^{-1}(\mathbb{K}(X))$  of  $A$  coincides with the set  $\{a \in A \mid \Phi(a) \in \mathbb{K}(X^\circ)\}$ . Thus, for each  $x \in I$ , we get  $\Psi(x) - 0 \oplus \Phi(x) = x \oplus 0 \in B$ . Thus,  $B$  contains  $I \oplus 0$  as a closed ideal, and we identify  $I$  with its copy inside  $B$ .

The next lemma will be used later.

**Lemma 4.1.4.** *Let  $Z$  be a Hilbert  $D$ -module. If an element  $x \in \mathbb{L}(Z)$  satisfies  $x \otimes 1 \in \mathbb{K}(Z \otimes_D A)$ , then  $x$  must be compact.*

*Proof.* We identify  $Z \otimes_D A \otimes_A X$  with  $Z \otimes_D X$ . Let  $\pi: \mathbb{L}(Z \otimes_D A) \rightarrow \mathbb{L}(Z \otimes_D X)$  be the  $*$ -homomorphism defined by  $\pi(y) = y \otimes 1_X$ . Since  $e_D$  commutes with  $\phi_X(D)$ , the projection  $1_Z \otimes e_D$  from  $Z \otimes_D X$  onto  $Z \otimes_D \xi_0 D$  belongs to  $\mathbb{L}(Z \otimes_D X)$ . Let  $\varphi: \mathbb{L}(Z \otimes_D X) \rightarrow \mathbb{L}(Z) \cong \mathbb{L}(Z \otimes_D \xi_0 D)$  be the compression by  $1_Z \otimes e_D$ . Then, one has  $\varphi(\pi(\theta_{\xi \otimes a, \eta \otimes b})) = \theta_{\xi, \eta E(ba^*)}$  for  $\xi, \eta \in Z$  and  $a, b \in A$ , and hence  $\varphi \circ \pi(\mathbb{K}(Z \otimes_D A)) \subset \mathbb{K}(Z)$ . Since  $x = \varphi \circ \pi(x \otimes 1_A)$  holds, we are done.  $\square$

Set  $\mathcal{B} := \{a \oplus (\phi_X(a) + K) \in A \oplus \mathbb{L}(X) \mid a \in A, K \in \mathbb{K}(X)\}$ . Then, we have the matrix representations:

$$\mathbb{K}(X) = \begin{bmatrix} \mathbb{K}(X^\circ) & X^\circ \\ (X^\circ)^* & D \end{bmatrix} \subset \mathcal{B} = \begin{bmatrix} B & X^\circ \\ (X^\circ)^* & D \end{bmatrix}. \quad (4)$$

Note that the semisplit exact sequence (3) is a “corner” of the split exact sequence

$$0 \longrightarrow \mathbb{K}(X) \longrightarrow \mathcal{B} \longrightarrow A \longrightarrow 0.$$

**Remark 4.1.5.** Assume that the data  $(D \subset A, E)$  comes from an inclusion of discrete groups  $\Lambda \leq \Gamma$ . If  $\Lambda$  is a finite index subgroup of  $\Gamma$ , then one has  $\mathcal{B} = A \oplus \mathbb{K}(X)$ . When the index is infinite, one has  $\mathbb{K}(X) \cap \phi_X(A) = \{0\}$ . Thus,  $\mathcal{B}$  is isomorphic to  $C^*(A, e_D) \subset \mathbb{L}(X)$ , which is isomorphic to  $(\mathbb{C}1 + c_0(\Gamma/\Lambda)) \rtimes_{\text{red}} \Gamma$ .

**4.2. Cuntz–Pimsner algebras.** Let  $(A, E) = \bigstar_D(A_k, E_k)$  be the reduced amalgamated free product of  $\{(D \subset A_k, E_k)\}_{k \in \mathcal{I}}$ , and  $(B_k, \Psi_k)$  be the semisplit extension associated with  $(D \subset A_k, E_k)$ :

$$0 \longrightarrow \mathbb{K}(X_k^\circ) \longrightarrow B_k \longrightarrow A_k \longrightarrow 0.$$

Consider the unital embedding  $\Psi_D: D \rightarrow \prod_{k \in \mathcal{I}} B_k; d \mapsto (\Psi_k(d))_{k \in \mathcal{I}}$ , and the  $C^*$ -algebra  $B := \bigoplus_{k \in \mathcal{I}} B_k + \Psi_D(D)$ . We denote the support projection of  $B_k$  in  $B$  by  $1_{B_k}$  and set  $B_k^\perp = 1_{B_k}^\perp B$ . Define the  $C^*$ -correspondence  $\mathfrak{X}$  over  $B$  by  $\bigoplus_{k \in \mathcal{I}} X_k^\circ \otimes_D B_k^\perp$  with the left action  $\phi_{\mathfrak{X}}: B \rightarrow \mathbb{L}(\mathfrak{X})$  defined by the direct sum of

$$B_k \rightarrow \mathbb{L}(X_k^\circ \otimes_D B_k^\perp); \quad \Psi_k(a) + K \mapsto (\Phi_k(a) + K) \otimes 1_{B_k}^\perp$$

for  $a \in A_k$  and  $K \in \mathbb{K}(X_k^\circ)$ . Note that the ideal  $I_k = \phi_{X_k}^{-1}(\mathbb{K}(X_k)) \subset B_k$  is contained in  $\ker \phi_{\mathfrak{X}}$  (see Remark 4.1.3). We denote the vector  $\xi_k \otimes 1_{B_k}^\perp$  in  $\mathfrak{X}$  by  $\xi_{k\bar{k}}$ .

**Remark 4.2.1.** In the case when  $\mathcal{I} = \{1, 2\}$ , we have much simpler descriptions:  $B = B_1 \oplus B_2$  and  $\mathfrak{X} = (X_1^\circ \otimes_D B_2) \boxplus (X_2^\circ \otimes_D B_1)$ .

**Lemma 4.2.2.** *It follows that  $J_{\mathfrak{X}} = \bigoplus_{k \in \mathcal{I}} \mathbb{K}(X_k^\circ)$ .*

*Proof.* We first prove the inclusion  $(\subset)$ : Take  $x \in J_{\mathfrak{X}}$  arbitrarily. By definition of  $B$ , there exist  $a_k \in A_k, d \in D$  and  $K_k \in \mathbb{K}(X_k^\circ)$  such that  $x = (\Psi_k(a_k + d) + K_k)_{k \in \mathcal{I}}$ . For each  $k \in \mathcal{I}$ , by Lemma 4.1.4 we have  $\Phi_k(a_k + d) \in \mathbb{K}(X_k^\circ)$ , and so  $a_k + d \in I_k$ . Since  $I_k \subset \ker \phi_{\mathfrak{X}}$  and  $J_{\mathfrak{X}} \subset (\ker \phi_{\mathfrak{X}})^\perp$ , we obtain  $(\Psi_k(a_k + d) + K_k)((a_k + d) \oplus 0) = 0$ , implying  $a_k + d = 0$ . Since  $k \in \mathcal{I}$  is arbitrary and  $(a_k)_k \in \bigoplus_k A_k$ , we conclude  $a_k = d = 0$  for all  $k \in \mathcal{I}$ , and so  $x = (K_k)_k \in \bigoplus_k \mathbb{K}(X_k^\circ)$ . The opposite inclusion follows from that  $\phi_{\mathfrak{X}}(\theta_{a\xi_k, b\xi_{k\bar{k}}}) = \theta_{a\xi_{k\bar{k}}, b\xi_{k\bar{k}}}$  for  $a, b \in A_k^\circ$  and  $k \in \mathcal{I}$ .  $\square$

**Proposition 4.2.3.** *There exists a universal covariant representation  $(\pi, t)$  of  $\mathfrak{X}$  on  $\Delta\mathbf{T}(A, E)$  such that  $\pi(\Psi_k(a)) = P_k a P_k$  for  $a \in A_k$ ,  $t(b\xi_{k\bar{k}}) = b P_k^\perp$  for  $b \in A_k^\circ$  and  $k \in \mathcal{I}$ , and  $C^*(\pi, t) = \Delta\mathbf{T}(A, E)$ .*

*Proof.* For each  $k \in \mathcal{I}$ , the operator  $P_k a P_k$  on  $P_k Y$  is unitarily equivalent to  $a \oplus \Phi_k(a) \otimes 1$  on  $A_k \oplus X_k^\circ \otimes_D X'$  for some injective  $D$ - $\prod_{j \in \mathcal{I}} A_j$   $C^*$ -correspondence  $X'$  by Remark 3.2.4 (ii). Thus, there exists an injective  $*$ -homomorphism  $\pi: B \rightarrow \Delta\mathbf{T}(A, E)$  such that  $\pi(\Psi_k(a)) = P_k a P_k$  for  $a \in A_k$ . For any  $k, j \in \mathcal{I}$ ,  $a \in A_k^\circ$ ,  $b \in A_j$ ,  $x \in B_k^\perp$  and  $y \in B_j^\perp$ , we have

$$(a P_k^\perp \pi(x))^* (b a P_j^\perp \pi(y)) = \delta_{k,j} \pi(x)^* E(a^* b) P_k^\perp \pi(y) = \delta_{k,j} \pi(x^* \Psi_D(E(a^* b)) y) = \delta_{k,j} \pi(\langle a\xi_{k\bar{k}}, b\xi_{k\bar{k}} y \rangle)$$

by Proposition 3.2.3. Thus, the map  $t: \mathfrak{X} \ni a\xi_{k\bar{k}} x \mapsto a P_k^\perp \pi(x) \in \Delta\mathbf{T}(A, E)$  defines an isometric right  $B$ -module map. Also, for any  $c \in A_k$  we have

$$t(\Phi_k(c) a\xi_{k\bar{k}}) = t((ca)^\circ \xi_k) = (ca)^\circ P_k^\perp = P_k c P_k a P_k^\perp = \pi(\Psi_k(c)) t(a\xi_{k\bar{k}}).$$

by Proposition 3.2.3 again. Thus, the pair  $(\pi, t)$  is an injective representation of  $\mathfrak{X}$ . The covariance of  $(\pi, t)$  follows from that

$$\pi(\theta_{a\xi_k, b\xi_k}) = \pi(\Psi_k(ab^*) - \Psi_k(a)\Psi_k(b^*)) = P_k a (1 - P_k) b^* P_k = t(a\xi_{k\bar{k}}) t(b\xi_{k\bar{k}})^*$$

for  $a, b \in A_k^\circ$  with Lemma 4.2.2.

For each  $n \in \mathbb{N}$ , let  $Q_n$  be the projection in  $\mathbb{L}(Y)$  corresponding to the subspace (see Remark 3.2.4 (ii))

$$\bigoplus_{k \in \mathcal{I}} \bigoplus_{\substack{\iota \in \mathcal{I}_{n+1} \\ \iota(n+1)=k}} X_{\iota(1)}^\circ \otimes_D \cdots \otimes_D X_{\iota(n)}^\circ \otimes_D A_k \quad \subset \quad \bigoplus_{k \in \mathcal{I}} X(r, k) \otimes_D A_k.$$

Then, letting  $U_z := e_{A_1} + e_{A_2} + \bigoplus_{n \geq 1} z^n Q_n$  for  $z \in \{w \in \mathbb{C} \mid |w| = 1\}$  we have  $\text{Ad } U_z(\pi(x)) = \pi(x)$  for  $x \in B$  and  $\text{Ad } U_z(t(\xi)) = z t(\xi)$  for  $\xi \in \mathfrak{X}$ . Therefore, the universality of  $(\pi, t)$  follows from the gauge-invariant uniqueness theorem [Ka, Theorem 6.4]. Finally,  $\Delta\mathbf{T}(A, E) \subset C^*(\pi, t)$  follows from the decomposition  $a = P_k a P_k + P_k a^\circ P_k^\perp + P_k^\perp a^\circ P_k + E(a) P_k^\perp$  for  $a \in A_k$ .  $\square$

**4.3. Toeplitz extensions.** We next see that the Toeplitz extension

$$0 \longrightarrow \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) \longrightarrow \mathcal{T}(\mathfrak{X}) \longrightarrow \mathcal{O}(\mathfrak{X}) \longrightarrow 0$$

is semisplit. Let  $\sigma_k: \Delta\mathbf{T}(A, E) \rightarrow \mathbb{L}(X^{(k)})$  be as in Remark 3.2.4 (i) and set  $(X^{\mathcal{I}}, \sigma) := \bigoplus_{k \in \mathcal{I}} (X^{(k)}, \sigma_k)$ . Then,  $(\tilde{\pi}, \tilde{t}) := (\sigma \circ \pi, \sigma \circ t)$  is also a representation of  $\mathfrak{X}$ . We denote the GNS vector in  $X^{(k)}$  by  $\xi_0^{(k)}$ . We fix a fixed-point free bijection  $\tau$  on  $\mathcal{I}$ . To simplify the notation, we will write  $\tau(k) = k + 1$  for  $k \in \mathcal{I}$ . Let  $\tilde{Q} \in \mathbb{L}(X^{\mathcal{I}})$  be the projection onto  $\bigoplus_{k \in \mathcal{I}} P_{(\tau, k)}^{\perp} X^{(k+1)}$ . Since  $\tilde{\pi}(B)$  commutes with  $\tilde{Q}$  and  $(1 - \tilde{Q})\tilde{t}(\mathfrak{X})\tilde{Q} = \{0\}$  holds, the pair  $(\pi', t') := (\tilde{\pi}(\cdot)\tilde{Q}, \tilde{t}(\cdot)\tilde{Q})$  is a representation of  $\mathfrak{X}$  on  $\mathbb{L}(QX^{\mathcal{I}})$ .

**Proposition 4.3.1.** *The representation  $(\Pi, T) := (\pi \oplus \pi', t \oplus t')$  of  $\mathfrak{X}$  on  $\Delta\mathbf{T}(A, E) \oplus \mathbb{L}(QX^{\mathcal{I}})$  is universal.*

*Proof.* It is clear that  $(\Pi, T)$  is an injective representation of  $\mathfrak{X}$  admitting a gauge action. Thus, we only have to check that  $\Pi(J_{\mathfrak{X}}) \cap \psi_T(\mathbb{K}(\mathfrak{X})) = \{0\}$  by [Ka, Proposition 3.3, Theorem 6.2]. Assume that an element  $x \in \mathbb{K}(X_k^{\circ})$  enjoys  $\Pi(x) \in \psi_T(\mathbb{K}(\mathfrak{X}))$ . Since  $t'(\xi)t'(\eta)^*$  vanishes on  $\bigoplus_{j \in \mathcal{I}} X_j^{(j+1)\circ}$  for all  $\xi, \eta \in \mathfrak{X}$ , so does  $\pi'(x)$ . Since the action of  $\pi'(x)$  on  $X_k^{(k+1)\circ}$  is unitarily equivalent to  $x \in \mathbb{L}(X_k^{\circ})$ , we have  $x = 0$ .  $\square$

Since  $(\Pi, T)$  is universal, there is a surjective  $*$ -homomorphism  $p: C^*(\Pi, T) \rightarrow C^*(\pi, t)$ . By [Ka, Proposition 4.6], Lemma 4.2.2 and the covariance of  $(\pi, t)$ , the kernel of  $p$  is generated by

$$\begin{aligned} \{\Pi(x) - \psi_T(\phi_{\mathfrak{X}}(x)) \mid x \in J_{\mathfrak{X}}\} &= 0 \oplus \overline{\text{span}}\{\pi'(\theta_{a\xi_k, b\xi_k}) - t'(a\xi_k)t'(b\xi_k)^* \mid k \in \mathcal{I}, a, b \in A_k^{\circ}\} \\ &= 0 \oplus \overline{\text{span}}\{Q\tilde{t}(a\xi_{k\bar{k}})(1 - Q)\tilde{t}(b\xi_{k\bar{k}})Q \mid k \in \mathcal{I}, a, b \in A_k^{\circ}\}. \end{aligned}$$

**Lemma 4.3.2.** *For any  $\xi, \eta \in \mathfrak{X}$ , the operator  $Q\tilde{t}(\xi)(1 - Q)\tilde{t}(\eta)^*Q$  belongs to the closed linear span of  $\{\pi'(x) - \psi_{t'}(\phi_{\mathfrak{X}}(x)) \mid x \in J_{\mathfrak{X}}\}$ .*

*Proof.* We may assume that  $\xi = a_k\xi_{k\bar{k}}x$  and  $\eta = a_l\xi_{l\bar{l}}y$  for some  $k, l \in \mathcal{I}$ ,  $a_k \in A_k^{\circ}$ ,  $a_l \in A_l^{\circ}$ ,  $x \in B_k^{\perp}$  and  $y \in B_l^{\perp}$ . Notice that if  $k = l$  and  $x, y \in \Psi_D(D)$ , then we are done. We observe that  $Q\tilde{t}(\xi)(1 - Q)\tilde{t}(\eta)^*Q$  is supported on  $\bigoplus_{j \in \mathcal{I}} X_j^{(j+1)\circ}$ . Fix  $j \in \mathcal{I}$  and  $a_j \in A_j^{\circ}$  and take  $b_{j+1} \in A_{j+1}$  and  $K \in \mathbb{K}(X_{j+1}^{\circ})$  in such a way that the  $(j+1)$ -th entry of  $xy^* \in B$  equals  $\Psi_{j+1}(b_{j+1}) + K$ . We then have

$$\begin{aligned} &Q\tilde{t}(a_k\xi_{k\bar{k}}x)(1 - Q)\tilde{t}(a_l\xi_{l\bar{l}}y)^*Qa_j\xi_0^{(j+1)} \\ &= \delta_{l,j}Q\tilde{t}(a_k\xi_{k\bar{k}})b_{j+1}\xi_0^{(j+1)}E(a_l^*a_j) \quad (\because a_j\xi_0^{(j+1)} = \tilde{t}(a_j\xi_{j\bar{j}})\xi_0^{(j+1)}) \\ &= \delta_{l,j}Q(a_k\xi_0 \otimes b_{j+1}^{\circ}\xi_0^{(j+1)} + a_k\xi_0^{(j+1)}E(b_{j+1}))E(a_l^*a_j) \\ &= \delta_{k,j}\delta_{l,j}a_kE(b_{j+1})\xi_0^{(j+1)}E(a_l^*a_j) \quad (\because \text{by definition of } Q) \\ &= \delta_{l,j}Q(a_kE(b_{j+1})\xi_0^{(j+1)})E(a_l^*a_j) \\ &= \delta_{l,j}Q\tilde{t}(a_kE(b_{j+1})\xi_{k\bar{k}})\xi_0^{(j+1)}E(a_l^*a_j) \\ &= \delta_{k,l}Q\tilde{t}(a_kE(b_{j+1})\xi_{k\bar{k}})(1 - Q)\tilde{t}(a_l\xi_{l\bar{l}})^*Qa_j\xi_0^{(j+1)} \end{aligned}$$

which completes the proof.  $\square$

**Proposition 4.3.3.** *Let  $\theta: \mathbb{L}(X^{\mathcal{I}}) \rightarrow \mathbb{L}(QX^{\mathcal{I}})$  be the compression by  $Q$  and set  $\Theta := \text{id} \oplus (\theta \circ \sigma): \Delta\mathbf{T}(A, E) \rightarrow \Delta\mathbf{T}(A, E) \oplus \mathbb{L}(QX^{\mathcal{I}})$ . Then,  $\Theta$  maps into  $C^*(\Pi, T)$  and defines a UCP cross section of the Toeplitz extension.*

*Proof.* It suffices to show that  $\Theta(t^{m+1}(\xi \otimes \xi')t^{n+1}(\eta \otimes \eta')^*) - T^{m+1}(\xi \otimes \xi')T^{n+1}(\eta \otimes \eta')^*$  is in  $\ker p$  for  $m, n \in \mathbb{N} \cup \{0\}$ ,  $\xi \in \mathfrak{X}^{\otimes m}$ ,  $\eta \in \mathfrak{X}^{\otimes n}$  and  $\xi', \eta' \in \mathfrak{X}$ . Indeed, one has

$$\Theta(t^{m+1}(\xi \otimes \xi')t^{n+1}(\eta \otimes \eta')^*) - T^{m+1}(\xi \otimes \xi')T^{n+1}(\eta \otimes \eta')^*$$



$$\begin{aligned}
&= 0 \oplus Q\tilde{t}^{m+1}(\xi \otimes \xi')(1-Q)\tilde{t}^{n+1}(\eta \otimes \eta')^*Q \\
&= 0 \oplus t'^m(\xi)Q\tilde{t}(\xi')(1-Q)\tilde{t}(\eta')^*Qt'^m(\eta)^*.
\end{aligned}$$

Here the last equality follows from  $Q\tilde{t}^m(\xi)(1-Q)\tilde{t}(\xi')(1-Q) = 0$ . Thus, the assertion follows from the previous lemma.  $\square$

**Corollary 4.3.4.** *The  $C^*$ -algebra  $\Delta\mathbf{T}(A, E)$  is nuclear (resp. exact) if and only if both  $A_1$  and  $A_2$  have the same property. Moreover, it follows that  $\Lambda_{\text{cb}}(\Delta\mathbf{T}(A, E)) = \max\{\Lambda_{\text{cb}}(A_1), \Lambda_{\text{cb}}(A_2)\}$ .*

*Proof.* Assume that  $A_1$  and  $A_2$  are nuclear (resp. exact). By Proposition 4.1.1,  $B$  has the same property, and hence so does  $\mathcal{T}(\mathfrak{X})$  by [Ka, Theorem 7.1, Theorem 7.2]. Thus, the same holds for  $\Delta\mathbf{T}(A, E)$  by Proposition 4.2.3 and Proposition 4.3.3. The opposite implication follows from Proposition 3.2.3 (iv). The assertion for CBAP follows from [DySm].  $\square$

We note that the above corollary does not need the deep results that nuclearity and exactness pass to quotients ([CE, Ki1, Ki2]). Since exactness passes to subalgebras, we obtain a new proof of Dykema's result.

**Corollary 4.3.5** (Dykema [Dy, DySh]). *Let  $(A, B) = (A_1, E_1) *_D (A_2, E_2)$  be a reduced amalgamated free product. Then,  $A$  is exact if and only if so are  $A_1$  and  $A_2$ . Moreover, if both  $A_1$  and  $A_2$  are nuclear, then  $A$  is a subalgebra of the nuclear  $C^*$ -algebra  $\Delta\mathbf{T}(A, E)$ .*

**Corollary 4.3.6** (Ozawa [Oz2]). *Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product of discrete amenable groups. Then the canonical action of  $\Gamma$  on the Bowditch compactification of its associated Bass–Serre tree is amenable.*

**Remark 4.3.7.** Let  $\Delta\mathbf{T}(A, E)^\sim$  be as in Remark 3.2.2. Let  $B^\sim := B \oplus D \subset \prod_{k \in \mathcal{I}} B_k$  and consider the  $C^*$ -correspondence  $\mathfrak{X}^\sim := \bigoplus_{k \in \mathcal{I}} X_k^\circ \otimes_D (1_{B^\sim} - 1_{B_k})B^\sim$  over  $B^\sim$ . Then, one can show that  $\Delta\mathbf{T}(A, E)^\sim$  is isomorphic to  $\mathcal{O}(\mathfrak{X}^\sim)$  naturally and the Toeplitz extension of  $\mathfrak{X}^\sim$  is semisplit in the same manner as Proposition 4.2.3 and Proposition 4.3.3.

**4.4. Reduced amalgamated free products.** In the case when  $\mathcal{I} = \{1, 2\}$ , the  $C^*$ -algebra  $\Delta\mathbf{T}(A, E)$  is also a reduced amalgamated free product  $C^*$ -algebra. We extend  $E_{A_k}$  to  $\Delta\mathbf{T}(A, E)$  by using  $e_{A_k}$  and still denote it by  $E_{A_k}$ . Then, the composition  $(E_1 \oplus E_2) \circ (E_{A_1} \oplus E_{A_2})$  defines a conditional expectation  $\mathcal{E}: \Delta\mathbf{T}(A, E) \rightarrow DP_1 \oplus DP_2 \cong D \oplus D$ . Set  $\mathcal{B}_k := C^*(A_k, P_1, P_2) \subset \Delta\mathbf{T}(A, E)$ ,  $L_k = C^*(t(X_k^\circ), DP_k^\perp)$ , and  $\mathcal{E}_k := \mathcal{E}|_{\mathcal{B}_k}$  for  $k = 1, 2$ , here we write  $t(X_k^\circ 1_{B_k}^\perp) = t(X_k^\circ)$  for short. Then we have the following matrix representations

$$\mathcal{B}_k = \begin{bmatrix} \pi(B_k) & t(X_k^\circ) \\ t(X_k^\circ)^* & DP_k^\perp \end{bmatrix}, \quad L_k = \begin{bmatrix} \pi(\mathbb{K}(X_k^\circ)) & t(X_k^\circ) \\ t(X_k^\circ)^* & DP_k^\perp \end{bmatrix}$$

with respect to the decomposition  $Y = P_k Y \oplus P_k^\perp Y$ , and the inclusion  $L_k \subset \mathcal{B}_k$  is isomorphic to that in Eq. (4).

**Proposition 4.4.1.** *We have  $(\Delta\mathbf{T}(A, E), \mathcal{E}) \cong (\mathcal{B}_1, \mathcal{E}_1) *_D (\mathcal{B}_2, \mathcal{E}_2)$ .*

*Proof.* Let  $\sigma = \sigma_1 \oplus \sigma_2: \Delta\mathbf{T}(A, E) \rightarrow \mathbb{L}(X^{(1)} \boxplus X^{(2)})$  be as in the previous subsection. Then, the GNS representation associated with  $\mathcal{E}$  is  $(X^{(1)} \boxplus X^{(2)}, \sigma, \xi_0^{(1)} \oplus \xi_0^{(2)})$ . Since  $\Delta\mathbf{T}(A, E)$  is generated by  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , we only have to check the freeness condition that  $\mathcal{E}(x_m \cdots x_1) = 0$  for any  $m \in \mathbb{N}$ ,  $\iota \in \mathcal{I}_m$ , and  $x_i \in \mathcal{B}_{\iota(i)}^\circ$  for  $i = 1, \dots, m$ . Note that

$$\mathcal{B}_1^\circ = \begin{bmatrix} \pi(\Psi_1(A_1^\circ) + \mathbb{K}(X_1^\circ)) & t(X_1^\circ) \\ t(X_1^\circ)^* & 0 \end{bmatrix}, \quad \mathcal{B}_2^\circ = \begin{bmatrix} 0 & t(X_2^\circ)^* \\ t(X_2^\circ) & \pi(\Psi_2(A_2^\circ) + \mathbb{K}(X_2^\circ)) \end{bmatrix}.$$

Thus, we may assume that each  $x_i$  is of the form  $P_{\iota(i)} y_i P_{\iota(i)} + P_{\iota(i)}^\circ \pi(K_i) P_{\iota(i)}^\circ + P_{\iota(i)} z_i P_{\iota(i)}^\perp + P_{\iota(i)}^\perp w_i P_{\iota(i)}$  for  $y_i, z_i, w_i \in A_{\iota(i)}^\circ$  and  $K_i \in \mathbb{K}(X_{\iota(i)}^\circ)$ . When  $\iota(1) = 1$ , one has  $\sigma_1(x_m \cdots x_1) \xi_0^{(1)} = \phi_X(z_m \cdots z_2 y_1) \xi_0^{(1)}$  and  $\sigma_2(x_m \cdots x_1) \xi_0^{(2)} = \phi_X(z_m \cdots z_1) \xi_0^{(2)}$ , and thus  $\mathcal{E}(x_m \cdots x_1) = 0$ . The case when  $\iota(1) = 2$  is similar.  $\square$

Note that the inclusion map  $\kappa_k: A_k \hookrightarrow \mathcal{B}_k$  satisfies that  $\kappa_k: E_k = \mathcal{E}_k: \kappa_k$  for each  $k = 1, 2$ . Thus, the embedding  $A \subset \Delta\mathbf{T}(A, E)$  is compatible with respect to the reduce amalgamated free product structures.

## 5. BOUNDARY ACTIONS

Recall that  $I_k := \phi_{X_k}^{-1}(A_k) \cong \{x \oplus 0 \mid x \in I_k\}$  forms an ideal of  $B_k$  (see Remark 4.1.3). Thus,  $I := \bigoplus_{k \in \mathcal{I}} I_k$  forms an ideal of  $B$ . Note that the operators  $xe_{A_k}$  and  $P_k^\circ x P_k^\circ$  are in  $\Delta\mathbf{T}(A, E)$  for  $x \in I_k$ . By Cohen's factorization theorem,  $Y_k I_k = \{\xi x \mid \xi \in Y_k, x \in I_k\}$  is a closed submodule of  $Y_k$ , and  $\mathbb{K}(Y_k I_k) = \overline{\text{span}}\{\theta_{a\eta_k x, b\eta_k} \in \mathbb{L}(Y_k) \mid a, b \in A, x \in I_k\}$  forms an ideal of  $\mathbb{L}(Y_k)$ . Thus,  $\bigoplus_{k \in \mathcal{I}} \mathbb{K}(Y_k I_k)$  is a closed ideal of  $\Delta\mathbf{T}(A, E)$  generated by  $\{xe_{A_k} \mid x \in I_k \text{ for } i \in \mathcal{I}\}$ .

**Definition 5.2.** The  $C^*$ -algebra  $\partial\mathbf{T}(A, E)$  is defined to be the quotient  $\Delta\mathbf{T}(A, E) / \bigoplus_{k \in \mathcal{I}} \mathbb{K}(Y_k I_k)$ .

In the group case,  $\partial\mathbf{T}(A, E)$  is indeed isomorphic to the reduced crossed product arising from the boundary action.

**Proposition 5.3.** *Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product and  $(A, E) = (A_1, E_1) *_D (A_2, E_2)$  be the corresponding reduced amalgamated free product. Then, there is a dichotomy:*

- $\Lambda$  is a finite index subgroup of  $\Gamma_k$  if and only if the ideal  $I_k$  coincides with  $A_k$ .
- $\Lambda$  is an infinite index subgroup of  $\Gamma_k$  if and only if the ideal  $I_k$  is  $\{0\}$ .

Also, via the isomorphism between  $\Delta\mathbf{T}(A, E)$  and  $C(\Delta\mathbf{T}) \rtimes_{\text{red}} \Gamma$ , the ideal  $\bigoplus_{k=1,2} \mathbb{K}(Y_k I_k)$  corresponds to  $(c_0(\mathbf{V}) \cap C(\mathbf{T})) \rtimes_{\text{red}} \Gamma$ .

*Proof.* If  $\Lambda$  is of finite index in  $\Gamma_k$ , then  $\mathbb{L}(X_k) = \mathbb{K}(X_k)$  holds, so  $I_k = A_k$ . Thus, for the first assertion, it is enough to show that if  $I_k \neq \{0\}$ , then one has  $[\Gamma_k : \Lambda] < \infty$ . We observe that  $\mathbb{K}(X_k) \cong \{K \otimes 1 \mid K \in \mathbb{K}(X_k)\} \subset \mathbb{L}(X_k \otimes_D A_k)$  is isomorphic to  $c_0(\Gamma_k/\Lambda) \rtimes_{\text{red}} \Gamma_k \subset \mathbb{L}(\ell^2(\Gamma_k/\Lambda) \otimes A_k)$ . Thus, if there exists a nonzero positive element  $x$  in  $I_k$ , then  $c_0(\Gamma_k/\Lambda) \rtimes_{\text{red}} \Gamma_k$  must be unital, and hence  $\Gamma_k/\Lambda$  is finite. The second assertion follows from that the projection  $e_{A_k} \otimes 1$  corresponds to the projection  $\delta_{\Gamma_k} \otimes 1$  in  $\ell^\infty(\Gamma_k) \otimes 1 \subset \mathbb{L}(\ell^2(\Gamma/\Gamma_k) \otimes A)$ .  $\square$

Let  $q: B \rightarrow B/I$  and  $\tilde{q}: \Delta\mathbf{T}(A, E) \rightarrow \partial\mathbf{T}(A, E)$  be the quotient maps and consider the pushout  $\mathfrak{X}_q := \mathfrak{X} \otimes_B (B/I)$ . Since  $I \subset \ker \phi_{\mathfrak{X}}$  holds, we have a left action  $\phi_{\mathfrak{X}_q}: B/I \rightarrow \mathbb{L}(\mathfrak{X}_q)$  such that  $\phi_{\mathfrak{X}_q} \circ q = \phi_{\mathfrak{X}} \otimes 1_{B/I}$ .

**Proposition 5.4.** *Assume that  $\Phi_k: D \rightarrow \mathbb{L}(X_k^\circ)$  defines an injective  $*$ -homomorphism for all  $k \in \mathcal{I}$ . Then, the following hold true:*

- (i) *the left action  $\phi_{\mathfrak{X}_q}$  is injective and  $J_{\mathfrak{X}_q} = q(\bigoplus_{k \in \mathcal{I}} \mathbb{K}(X_k^\circ))$  holds.*
- (ii) *the restriction of  $\tilde{q}$  to  $A$  is injective,*
- (iii)  *$\partial\mathbf{T}(A, E)$  is isomorphic to the Cuntz–Pimsner algebra of  $\mathfrak{X}_q$ .*

*Proof.* It follows from the assumption that  $\Psi_k(D) \cap I_k = \{0\}$  for every  $k \in \mathcal{I}$ . Thus, if  $x \in B_k$  enjoys that  $\phi_{\mathfrak{X}_q}(q(x)) = 0$ , then  $x = a \oplus 0 \in A_k \oplus \mathbb{L}(X_k^\circ)$  for some  $a \in I_k$ , and so  $q(x) = 0$ . Thus,  $\phi_{\mathfrak{X}_q}$  is injective and we have  $J_{\mathfrak{X}_q} = q(\bigoplus_{k \in \mathcal{I}} \mathbb{K}(X_k^\circ))$ .

To see the assertion (ii), we actually prove that  $\phi_{Y_k}(A) \cap \mathbb{K}(Y_k) = \{0\}$  for  $k \in \mathcal{I}$ . On the contrary, we suppose that there exists an element  $a \in \phi_{Y_k}^{-1}(\mathbb{K}(Y_k))$  with  $\|a\| = 1$  for some  $k \in \mathcal{I}$ . We may omit  $\phi_{Y_k}$  and assume that  $A_k \subset \mathbb{L}(Y_k)$ . For each  $n \geq 1$ , let  $Q_n \in \mathbb{L}(Y_k)$  be the projection corresponding to the submodule

$$\bigoplus_{\substack{\iota \in \mathcal{I}_n \\ \iota(n) \neq k}} X_{\iota(1)}^\circ \otimes_D \cdots \otimes X_{\iota(n)}^\circ \otimes_D A_k \subset X(r, k) \otimes_D A_k$$

via the unitary  $S_k: X(r, k) \otimes_D A_k \rightarrow Y_k$ . Then, we have  $\lim_{n \rightarrow \infty} \|aQ_n\| = 0$ . There exist  $m \in \mathbb{N}$ ,  $\iota \in \mathcal{I}_m$  with  $\iota(m) \neq k$  and a unit vector  $\zeta_1 \in \text{span } A_{\iota(1)}^\circ \cdots A_{\iota(m)}^\circ \eta_k$  such that  $\|a\zeta_1\| = \delta > 0$ . Then,  $\langle a\zeta_1, a\zeta_1 \rangle$  is a positive contraction in  $D$  such that  $\|\langle a\zeta_1, a\zeta_1 \rangle\| > \delta$ . Thus,

we can find a unit vector  $\zeta_2 \in X_{\iota(m+1)}^\circ \otimes_D X_{\iota(m+2)}^\circ$  with  $\iota(m) \neq \iota(m+1) \neq \iota(m+2) \neq k$  such that  $\|\langle \zeta_2, \langle a\zeta_1, a\zeta_1 \rangle \zeta_2 \rangle\| > \delta$ . This implies that  $\|aQ_{m+2}\| > \delta$ . Similarly, we can inductively show that  $\|aQ_{m+2n}\| > \delta$  for all  $n \in \mathbb{N}$ , a contradiction.

We prove (iii): Let  $(\pi, t)$  be as in Proposition 4.2.3. Since  $\pi(I) \subset \bigoplus_{k \in \mathcal{I}} \mathbb{K}(Y_k I_k)$  holds, there is a covariant representation  $(\bar{\pi}, \bar{t})$  of  $\mathfrak{X}_q$  on  $\partial \mathbf{T}(A, E)$  such that  $\bar{\pi} \circ q = \bar{q} \circ \pi$  and  $\bar{t}(a\xi_{k\bar{k}}) = \bar{q}(aP_k^\perp)$  for  $a \in A_k^\circ$ . Since  $\bigoplus_{k \in \mathcal{I}} \mathbb{K}(X_k^\circ)$  is gauge action invariant,  $(\bar{\pi}, \bar{t})$  admits a gauge action. Thus, it suffices to show that  $\bar{\pi}$  is injective, equivalently  $\pi(B) \cap \bigoplus_{k \in \mathcal{I}} \mathbb{K}(Y_k I_k) = \pi(I)$ . For any  $x = a \oplus b \in B_k \setminus I_k$ , the operator  $P_k^\circ \pi(x) P_k^\circ$  is nonzero. Since  $D$  acts on  $X_l^\circ$  faithfully for  $l \in \mathcal{I}$ ,  $P_k^\circ \pi(x) P_k^\circ$  is an infinite direct sum of faithful representations of  $b \in \mathbb{L}(X_k^\circ)$ . This implies that  $\pi(x) \notin \bigoplus_{l \in \mathcal{I}} \mathbb{K}(X_l^\circ)$ .  $\square$

In the case when  $\Gamma_1$  and  $\Gamma_2$  are finite, the quotient of  $C(\Delta \mathbf{T}) \rtimes_{\text{red}} \Gamma$  by  $c_0(\mathbf{V}) \rtimes_{\text{red}} \Gamma$  is isomorphic to  $C(\partial \mathbf{T}) \rtimes_{\text{red}} \Gamma$ . It was shown by Okayasu that  $C(\partial \mathbf{T}) \rtimes_{\text{red}} \Gamma$  is isomorphic to some Cuntz–Pimsner algebra ([Ok]). We will generalize Okayasu’s result to  $\partial \mathbf{T}(A, E)$  in the case when  $A_k = I_k$  for  $k \in \mathcal{I}$ . Set  $\mathcal{D} := \bigoplus_{k \in \mathcal{I}} D 1_{B_k} + \Psi_D(D) \cong (\mathbb{C} 1 + c_0(\mathcal{I})) \otimes D$  and  $\mathfrak{Y} := \bigoplus_{k \in \mathcal{I}} X_k^\circ \otimes_D 1_{B_k}^\perp \mathcal{D} \subset \mathfrak{X}$  with the left action  $\phi_{\mathfrak{Y}}(d) = \phi_{\mathfrak{X}}(d)|_{\mathfrak{Y}}$  for  $d \in \mathcal{D}$ .

**Proposition 5.5** (c.f. [Ok, Theorem 4.9]). *Assume that  $\Phi_k: D \rightarrow \mathbb{L}(X_k^\circ)$  is injective and  $I_k = A_k$  holds for every  $k \in \mathcal{I}$ . Then  $\partial \mathbf{T}(A, E)$  is isomorphic to the Cuntz–Pimsner algebra  $\mathcal{O}(\mathfrak{Y})$ .*

*Proof.* Let  $(\pi_0, t_0)$  be the representation of  $\mathfrak{Y}$  on  $\partial \mathbf{T}(A, E)$  given by  $\pi_0(d 1_{B_k}) = \bar{q}(d P_k)$  and  $t_0(a\xi_{k\bar{k}}) = \bar{q}(a P_k^\perp)$  for  $k \in \mathcal{I}$ ,  $d \in D$  and  $a \in B_k$ . Then  $(\pi_0, t_0)$  is injective and covariant and admits a gauge action. Thus, it suffices to show that  $A_k^\circ \subset C^*(\pi_0, t_0)$  for all  $k \in \mathcal{I}$ . Indeed, for any  $a \in A_k^\circ$  one has  $\bar{q}(a) = \psi_{t_0}(\phi_{\mathfrak{X}}(\Psi_k(a))) + t_0(a\xi_{k\bar{k}}) + t_0(a^* \xi_{k\bar{k}})^*$ .  $\square$

Note that  $\partial \mathbf{T}(A, E)$  is not necessarily simple even in the group case (see [Ok]). We close this section by giving a short proof of Ozawa’s result [Oz1] on nucleartiy of  $A$ .

**Corollary 5.6** (Ozawa). *Let  $(A, E) = (A_1, E_1) *_D (A_2, E_2)$  be a reduced amalgamated free product of nuclear  $C^*$ -algebras. If the image of the GNS-representation associated with  $E_1$  contains the Jones projection  $e_{D^1}^1$ , then  $A$  is nuclear.*

*Proof.* By assumption, there exists a projection  $p \in A_1$  such that  $\phi_{X_1}(p) = e_{D^1}^1$ . The direct computation shows that  $p = p e_{A_1} + P_2$  in  $\Delta \mathbf{T}(A, E)$ . Since  $p e_{A_1}$  is in  $\mathbb{K}(Y_1 I_1)$ , we have  $A + \mathbb{K}(Y_1 I_1) = \Delta \mathbf{T}(A, E)$ . Since  $A \cap \mathbb{K}(Y_1 I_1) = \{0\}$  holds, we have a split extension

$$0 \longrightarrow \mathbb{K}(Y_1 I_1) \longrightarrow \Delta \mathbf{T}(A, E) \longrightarrow A \longrightarrow 0.$$

Thus, the assertion follows from Corollary 4.3.4.  $\square$

## 6. KK-THEORY

In this section, we prove Theorem C and Theorem D. Throughout this section, we assume that  $A_1$  and  $A_2$  are separable. Let  $\phi = \phi_Y: A \hookrightarrow \Delta \mathbf{T}(A, E)$  be the inclusion map and set  $\rho_k := \pi \circ \Psi_k|_D: D \hookrightarrow D P_k \subset \Delta \mathbf{T}(A, E)$ . We denote by  $\bar{k}$  the unique element in  $\{1, 2\} \setminus \{k\}$ .

**Notation 6.7.** We use the following traditional notations:

- Let  $P$  be a separable  $C^*$ -algebra. For each  $k = 1, 2$ , we denote by  $(X_k^\circ)_*$  (resp.  $(X_k^\circ)^*$ ) the map  $KK^p(P, \mathbb{K}(X_k^\circ)) \rightarrow KK^p(P, D)$  (resp.  $KK^p(D, P) \rightarrow KK^p(\mathbb{K}(X_k^\circ), P)$ ) induced from the element  $[(X_k^\circ, \text{id}, 0)]$  in  $KK(\mathbb{K}(X_k^\circ), D)$ . This element is just the Kasparov product of the natural inclusion map  $\mathbb{K}(X_k^\circ) \hookrightarrow \mathbb{K}(X_k)$  and the inverse element of the map  $D \ni d \mapsto \phi_{X_k}(d) e_D \in \mathbb{K}(X_k)$ .
- For any  $C^*$ -algebra  $P$ , we set  $SP := C_0((0, 1), P)$  and  $CP := C_0([0, 1], P)$ , and set  $S = S\mathbb{C}$  and  $C = C\mathbb{C}$ . Also, we set  $S\varphi = \text{id}_S \otimes \varphi: SP \rightarrow SQ$  and  $\tau_S: KK^p(P, Q) \ni x \mapsto \text{id}_S \otimes x \in KK^p(SP, SQ)$ .
- For a  $*$ -homomorphism  $\varphi: P \rightarrow Q$ , the mapping cone of  $\varphi$  is defined by

$$C_\varphi = \{(x, f) \in P \oplus CQ \mid f(0) = \varphi(x)\}.$$

**Lemma 6.8.** *There exist  $\alpha \in KK(\Delta\mathbf{T}(A, E), A)$  and  $\delta \in KK(\Delta\mathbf{T}(A, E), D)$  such that  $(\phi \oplus \rho) \otimes_{\Delta\mathbf{T}(A, E)} (\alpha \oplus \beta) = \text{id}_A \oplus \text{id}_D$ .*

*Proof.* Let  $(Z, \phi_Z)$  and  $\tilde{\phi}_Z: \Delta\mathbf{T}(A, E) \rightarrow \mathbb{L}(Z)$  be as in Eq. (1). Define the isometry  $S: X \otimes_D A \rightarrow Z$  by

$$S = \begin{cases} S_1 \otimes 1 : X(r, 1) \otimes_D A \rightarrow Y_1 \otimes_{A_1} A; \\ S_2 \otimes 1 : X(r, 2)^\circ \otimes_D A \rightarrow Y_2^\circ \otimes_{A_2} A. \end{cases}$$

It follows from [Ha, Lemma 3.2] that  $S(\phi_X(a) \otimes 1) - \phi_Z(a)S$  is compact for  $a \in \Delta\mathbf{T}(A, E)$ . Since  $\tilde{\phi}_Z(P_1)S = S(\sigma_1(P_1) \otimes 1)$  holds, the triplet

$$(Z \oplus (X \otimes_D A), \tilde{\phi}_Z \oplus (\sigma_1 \otimes 1), \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix})$$

is a  $\Delta\mathbf{T}(A, E)$ - $A$  Kasparov bimodule and defines an element  $\alpha \in KK(\Delta\mathbf{T}(A, E), A)$ . Since  $\phi \otimes_{\Delta\mathbf{T}(A, E)} \alpha$  is implemented by the  $A$ - $A$  Kasparov bimodule

$$(Z \oplus (X \otimes_D A), \phi_Z \oplus (\phi_X \otimes 1), \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}),$$

we have  $\phi \otimes_{\Delta\mathbf{T}(A, E)} \alpha = \text{id}_A \in KK(A, A)$  by [Ha, Theorem 3.4]. Also, it follows from  $\tilde{\phi}_Z(P_1) = S(\sigma_1(P_1) \otimes 1)S^*$  that  $\rho_1 \otimes_{\Delta\mathbf{T}(A, E)} \alpha = 0$ .

Let  $\sigma_k: \Delta\mathbf{T}(A, E) \rightarrow \mathbb{L}(X^{(k)})$  be as in Remark 3.2.4 (i). Since  $\sigma_1 = \sigma_2 = \phi_X$  on  $A$  and  $\sigma_1(P_1) - \sigma_2(P_1) = e_D$  holds, the triplet  $(X^{(1)} \oplus X^{(2)}, \sigma_1 \oplus \sigma_2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$  is a  $\Delta\mathbf{T}(A, E)$ - $D$  Kasparov bimodule and the corresponding element  $\delta \in KK(\Delta\mathbf{T}(A, E), D)$  satisfies that  $\rho_1 \otimes_{\Delta\mathbf{T}(A, E)} \delta = \text{id}_D$  and  $\phi \otimes_{\Delta\mathbf{T}(A, E)} \delta = 0$ .  $\square$

Our proof of Theorem C is based on the six-term exact sequences induced from the semisplit Toeplitz extension. Thanks to the next lemma, we can assume that  $X_1^\circ$  and  $X_2^\circ$  are full, i.e.,  $\overline{\text{span}}\{E(a^*b) \mid a, b \in A_k^\circ\} = D$  holds for each  $k = 1, 2$ .

**Lemma 6.9.** *Let  $\varphi$  be a nondegenerate state on  $D$  and set  $\varphi_k := \varphi \circ E_k$  for  $k = 1, 2$ ,  $(\mathcal{T}, \omega)$  be the Toeplitz algebra with the vacuum state, and  $(\mathcal{A}_k, \tilde{\varphi}_k) = (A_k, \varphi_k) * (\mathcal{T}, \omega)$  be the reduced free product. Denote by  $F_k: \mathcal{A}_k \rightarrow D$  the composition of the canonical conditional expectation  $\mathcal{A}_k \rightarrow A_k$  and  $E_k: A_k \rightarrow D$  and by  $\mathcal{X}_k$  the GNS Hilbert  $C^*$ -module of  $F_k$ . Set  $(\mathcal{A}, F) = (\mathcal{A}_1, F_1) *_D (\mathcal{A}_2, F_2)$ . Then  $\mathcal{X}_k^\circ$  is full and the embedding maps  $A_k \hookrightarrow \mathcal{A}_k$  and  $A \hookrightarrow \mathcal{A}$  induce  $KK$ -equivalences.*

*Proof.* Let  $s$  be the unilateral shift generating  $\mathcal{T}$ . Then, one has  $F_k(s) = 0$  and  $F_k(s^*s) = 1$ , and thus  $\mathcal{X}_k^\circ$  is full. Let  $\mathcal{H}_k$  and  $\mathcal{H}$  be the  $C^*$ -correspondences over  $A_k$  and  $A$  associated with the UCP maps  $\varphi_k 1$  on  $A_k$  and  $\varphi 1$  on  $A$ , respectively. Then, it is well-known that  $(\mathcal{T}(\mathcal{H}_k), \varphi_k \circ E_{\mathcal{H}_k}) \cong (A_k, \varphi_k) * (\mathcal{T}, \omega)$ . Thus, the embedding  $A_k \hookrightarrow \mathcal{A}_k$  induces a  $KK$ -equivalence by [Pi2]. Similarly, by Speicher's theorem, we have

$$(\mathcal{A}, \varphi \circ F) \cong (A, \varphi \circ E) * (\mathcal{T}, \omega) * (\mathcal{T}, \omega) \cong (\mathcal{T}(\mathcal{H}), E_{\mathcal{H}}) *_A (\mathcal{T}(\mathcal{H}), E_{\mathcal{H}}) \cong (\mathcal{T}(\mathcal{H} \oplus \mathcal{H}), E_{\mathcal{H} \oplus \mathcal{H}}),$$

and thus  $A \hookrightarrow \mathcal{A}$  gives a  $KK$ -equivalence by [Pi2].  $\square$

**Remark 6.10.** Since  $\overline{\text{span}}\{E(ab) \mid a, b \in A_k^\circ\} + A_k^\circ$  is an ideal of  $A_k$ , the module  $X_k^\circ$  is full if  $A_k$  is simple. In the previous lemma, one can choose  $\mathcal{A}_k$  as simple  $C^*$ -algebra by replacing  $\mathcal{T}$  by the Cuntz algebra  $\mathcal{O}_\infty$  of countably many generators with the vacuum state  $\omega$ . Indeed,  $(\mathcal{O}_\infty, \omega)$  is isomorphic to the infinite free product  $\bigstar(\mathcal{T}, \omega)$ , and hence the reduced free product of  $A_k$  and  $\mathcal{O}_\infty$  is isomorphic to  $\mathcal{T}(\bigoplus_{\mathbb{N}} \mathcal{H}_k)$ , which is simple by [Ku].

Before the proof of Theorem C we give a short proof of the next proposition on Skandalis's  $K$ -nuclearity ([Sk2]). Note that this is also follows from the result in [FG1] that  $SA$  is  $KK$ -equivalent to the mapping cone of the diagonal embedding  $D \hookrightarrow A_1 \oplus A_2$ .

**Proposition 6.11.** *Reduced amalgamated free products of  $K$ -nuclear  $C^*$ -algebras over  $K$ -nuclear  $C^*$ -subalgebras are  $K$ -nuclear.*

*Proof.* Assume that  $A_1, A_2$  and  $D$  are  $K$ -nuclear. We may assume that  $X_1^\circ$  and  $X_2^\circ$  are full by Lemma 6.9. Since  $\mathbb{K}(X_k^\circ)$  is  $KK$ -equivalent to  $D$ , it is  $K$ -nuclear. It follows from [Sk2] that the semisplit extension  $B_k$  of  $A_k$  by  $\mathbb{K}(X_k^\circ)$  has the same property. These facts imply that  $\mathcal{T}(\mathfrak{X})$  and  $\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$  are  $K$ -nuclear, and thus  $\Delta\mathbf{T}(A, E) \cong \mathcal{O}(\mathfrak{X})$  is again  $K$ -nuclear by Proposition 4.2.3 and Proposition 4.3.3. Therefore,  $\phi \otimes_{\Delta\mathbf{T}(A, E)} \alpha$  is implemented by some nuclear Kasparov bimodule, and hence the  $K$ -nuclearity of  $A$  follows from Lemma 6.8.  $\square$

Now we use the identifications  $\Delta\mathbf{T}(A, E) = \mathcal{O}(\mathfrak{X})$  and  $C^*(\Pi, T) = \mathcal{T}(\mathfrak{X})$  by Proposition 4.2.3 and Proposition 4.3.1. We also identify the kernel of the quotient map  $p: \mathcal{T}(\mathfrak{X}) \rightarrow \mathcal{O}(\mathfrak{X})$  with  $\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$ . Since the Toeplitz extension is semisplit by Proposition 4.3.3, there is a six-term exact sequence for any separable  $C^*$ -algebra  $P$  ([Pi2, Ka]):

$$\begin{array}{ccccc} KK(P, J_{\mathfrak{X}}) & \xrightarrow{\iota_* - [\mathfrak{X}]} & KK(P, B) & \xrightarrow{\pi_*} & KK(P, \mathcal{O}(\mathfrak{X})) \\ \uparrow & & & & \downarrow \\ KK^1(P, \mathcal{O}(\mathfrak{X})) & \xleftarrow{\pi_*} & KK^1(P, B) & \xleftarrow{\iota_* - [\mathfrak{X}]} & KK^1(P, J_{\mathfrak{X}}), \end{array}$$

where  $[\mathfrak{X}]$  is induced from  $[(\mathfrak{X}, \phi_{\mathfrak{X}}|_{J_{\mathfrak{X}}}, 0)] \in KK(J_{\mathfrak{X}}, B)$ . We use the following inclusion maps:

$$\begin{aligned} \iota_{\Omega}: J_X &\rightarrow \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}); & x &\mapsto \Pi(x) - \psi_T(\phi_{\mathfrak{X}}(x)), \\ \iota^{\circ}: J_{\mathfrak{X}} &\rightarrow \mathbb{K}(X_1 \oplus X_2); & x &\mapsto x, \\ \iota_{\theta}: D \oplus D &\rightarrow \mathbb{K}(X_1 \oplus X_2); & d_1 \oplus d_2 &\mapsto \phi_{X_1}(d_1)e_D \oplus \phi_{X_2}(d_2)e_D. \end{aligned}$$

Note that  $\iota_{\Omega}$  and  $\iota_{\theta}$  induce  $KK$ -equivalences. Also,  $\iota^{\circ}$  induces a  $KK$ -equivalence if each  $X_k^\circ$  is full. Let  $\delta_p \in KK^1(\mathcal{O}(\mathfrak{X}), \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}))$  be the element corresponding to the Toeplitz extension (see [Sk, § 1]). Then, the connecting map  $KK^p(P, \mathcal{O}(\mathfrak{X})) \rightarrow KK^{p+1}(P, J_{\mathfrak{X}})$  is given by  $\delta_p \otimes_{\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})} (\iota_{\Omega})^{-1} \in KK^1(\mathcal{O}(\mathfrak{X}), J_{\mathfrak{X}})$ .

**Lemma 6.12.** *Assume that  $X_1^\circ$  and  $X_2^\circ$  are full. Then there is a cyclic exact sequence*

$$\begin{array}{ccccc} KK(P, D \oplus D) & \xrightarrow[\xi]{} & KK(P, A_1 \oplus A_2 \oplus D \oplus D) & \xrightarrow[\eta]{} & KK(P, \mathcal{O}(\mathfrak{X})) \\ \uparrow \partial & & & & \downarrow \partial \\ KK^1(P, \mathcal{O}(\mathfrak{X})) & \xleftarrow[\eta]{} & KK^1(P, A_1 \oplus A_2 \oplus D \oplus D) & \xleftarrow[\xi]{} & KK^1(P, D \oplus D), \end{array}$$

where  $\xi(x, y) = (-i_{1*}(y), -i_{2*}(x), x + y, x + y)$  and  $\eta = \phi_* \circ j_{1*} + \phi_* \circ j_{2*} + \rho_{1*} + \rho_{2*}$ . Also  $\partial$  is given by  $\delta_p \otimes_{\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})} (\iota_{\Omega})^{-1} \otimes_{J_{\mathfrak{X}}} \iota^{\circ} \otimes_{\mathbb{K}(X_1 \oplus X_2)} (\iota_{\theta})^{-1} \in KK^1(\mathcal{O}(\mathfrak{X}), D \oplus D)$ .

*Proof.* We set  $F^p := KK^p(P, -)$  for  $p = 0, 1$ . Since  $X_k^\circ$  is full, the  $C^*$ -algebra  $B_k$  is a full corner of  $\mathcal{B}_k$ , so the inclusion map induces a  $KK$ -equivalence by [Br]. Since  $\mathcal{B}_k$  is a split extension of  $A_k$  by  $L_k$ , one has a  $KK$ -equivalence between  $\mathcal{B}_k$  and  $A_k \oplus L_k$ . Also, the embedding  $D \hookrightarrow L_k; d \mapsto \phi_{X_k}(d)e_D$  gives a  $KK$ -equivalence. Let  $\mu_k: F(B_k) \rightarrow F(A_k \oplus D)$  be the isomorphism given by the compositions of these  $KK$ -equivalences.

Note that  $[\mathfrak{X}]$  is the direct sum of two maps  $(\Psi_{\overline{k}} \circ i_{\overline{k}})_* \circ (X_k^\circ)_*: F(\mathbb{K}(X_k^\circ)) \rightarrow F(B_{\overline{k}})$  for  $k = 1, 2$ . Thus, we have the following commuting diagrams in which vertical arrows are isomorphisms:

$$\begin{array}{ccccc} F(\mathbb{K}(X_k^\circ)) & \xrightarrow[\iota_*]{} & F(B_k) & & F(\mathbb{K}(X_k^\circ)) \xrightarrow[(\Psi_{\overline{k}} \circ i_{\overline{k}})_* \circ (X_k^\circ)_*]{} F(B_{\overline{k}}) \\ \downarrow (X_k^\circ)_* & & \downarrow \mu_k & & \downarrow (X_k^\circ)_* \\ F(D) & \xrightarrow[(0,1)]{} & F(A_k \oplus D), & & F(D) \xrightarrow[(i_{\overline{k}*} \times (-1))]{\phantom{}} F(A_{\overline{k}} \oplus D). \end{array}$$

Therefore, the assertion follows from  $(X_1^\circ)_* \oplus (X_2^\circ)_* = (\iota_{\theta})_*^{-1} \circ (\iota^{\circ})_*$  as a map from  $F(J_{\mathfrak{X}})$  onto  $F(D \oplus D)$ .  $\square$

It follows from the previous two lemmas that the sequence for  $KK^p(P, -)$  in Theorem C is exact at  $A_1 \oplus A_2$  and  $A$ . Indeed, it follows from  $j_{1*} \circ i_{1*} = (\rho_1 \oplus \rho_2)_*$  that  $\text{Im } \eta \subset \text{Im } \phi_* + \rho_{1*}$ . Thus, it follows from the injectivity of  $\phi_*$  and Lemma 6.8 that  $\ker(j_{1*} + j_{2*}) = \text{Im}(i_{1*}, -i_{2*})$  and  $\ker \partial \circ \phi_* = \text{Im}(j_{1*} + j_{2*})$ . Also, we have  $\partial(\phi_*(F(A))) \subset \ker \xi = \{(x, -x) \mid x \in \ker((i_1)_*, (i_2)_*)\}$ .

To see the exactness at  $D$ , we consider the element  $u = \phi \otimes_{\mathcal{O}(\mathfrak{X})} \delta_p \otimes_{\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})} \otimes (\iota_{\Omega})^{-1} \otimes_{J_{\mathfrak{X}}} \iota^{\circ}$  in  $KK^1(A, \mathbb{K}(X_1 \boxplus X_2))$ . Then, the exactness at  $D$  follows once we prove the following inclusion

$$(S\iota_{\theta})_*(\{(-x, x) \in F^1(SD \oplus SD) \mid x \in \ker Si_*\}) \subset \tau_S(u)_*(F(SA))$$

for any separable  $C^*$ -algebra  $P$ . For this, we use the mapping cone  $C_i$  of the diagonal map  $i = i_1 \oplus i_2: D \hookrightarrow A_1 \oplus A_2$ . Let  $G: C_i \rightarrow SA$  be Germain's  $*$ -homomorphism defined by

$$G(d, f_1 \oplus f_2)(t) = \begin{cases} f_1(1 - 2t) & t \in (0, 1/2], \\ f_2(2t - 1) & t \in [1/2, 1). \end{cases}$$

Then, the next lemma obviously implies the above inclusion.

**Lemma 6.13.** *Assume that  $X_1^{\circ}$  and  $X_2^{\circ}$  are full. Then, we have the inclusion  $(S\iota_{\theta})_*(\{(x, -x) \in F^1(SD \oplus SD) \mid x \in \ker Si_*\}) \subset \tau_S(u)_* \circ G_*(F(C_i))$ .*

*Proof.* Let  $C_{\Theta}$  and  $C_{\rho}$  be the mapping cones of  $D \ni d \mapsto \Theta(d) \oplus \Theta(d) \in C^*(\Theta(A_1) \oplus \Theta(A_2))$  and  $D \ni d \mapsto \rho_1(d) \oplus \rho_2(d) \in \mathcal{B}_1 \oplus \mathcal{B}_2$ , respectively. We observe that  $\Theta(ab^*) - \Theta(a)\Theta(b^*) = \iota_{\Omega}(\theta_{a\xi_k, b\xi_k})$  holds for all  $a, b \in A_k^{\circ}$  and  $k = 1, 2$ . Hence, we have  $C^*(\Theta(A_k)) = \Theta(A_k) + \iota_{\Omega}(\mathbb{K}(X_k^{\circ}))$ . Define embedding maps  $u: SJ_{\mathfrak{X}} \rightarrow C_{\Theta}$  and  $\iota_{\boxplus}: S\mathbb{K}(X_1 \boxplus X_2) \rightarrow C_{\rho}$  by formula

$$(f_1 \oplus f_2) \mapsto (t \mapsto (f_1(1 - t) \oplus f_2(t))),$$

and define  $\tilde{G}: C_{\Theta} \rightarrow \mathcal{ST}(\mathfrak{X})$  in a similar way to  $G$ . Note that the composition  $\tilde{G} \circ v: SJ_{\mathfrak{X}} \rightarrow S\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$  is homotopic to  $S\iota_{\Omega}$ . We observe that the restriction of the UCP map  $E_{\mathfrak{X}}(\cdot)1_{B_k}: \mathcal{T}(\mathfrak{X}) \rightarrow B_k$  to  $C^*(\Theta(A_k))$  gives a surjective  $*$ -homomorphism onto  $B_k \subset \mathcal{B}_k$  for  $k = 1, 2$ , and they induce a  $*$ -homomorphism  $q: C_{\Theta} \rightarrow C_{\rho}$ . Then, we have the commuting diagram of semisplit exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S\mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) & \longrightarrow & \mathcal{ST}(\mathfrak{X}) & \xrightarrow{Sp} & \mathcal{SO}(\mathfrak{X}) \longrightarrow 0 \\ & & \uparrow & & \tilde{G} \uparrow & & S\phi \circ G \uparrow \\ 0 & \longrightarrow & SJ_{\mathfrak{X}} & \xrightarrow{v} & C_{\Theta} & \longrightarrow & C_i \longrightarrow 0 \\ & & S\iota^{\circ} \downarrow & & q \downarrow & & \downarrow \\ 0 & \longrightarrow & S\mathbb{K}(X_1 \boxplus X_2) & \xrightarrow{\iota_{\boxplus}} & C_{\rho} & \xrightarrow{r} & C_i \longrightarrow 0. \end{array}$$

Thus,  $G \otimes_{SA} \tau_S(u) = \delta_r$  by [Sk, Lemma 1.5]. Since  $\delta_r$  gives the connecting map in the six-term exact sequence induced from  $r$ , the image of the map  $G \otimes_{SA} \tau_S(u)_*: F(C_i) \rightarrow F^1(S\mathbb{K}(X_1 \boxplus X_2))$  is just the kernel of  $(\iota_{\boxplus})_*: F^1(S\mathbb{K}(X_1 \boxplus X_2)) \rightarrow F^1(C_{\rho})$ . So, we only have to check that  $(\iota_{\boxplus})_* \circ (S\iota_{\theta})_*(x, -x) = 0$  for  $x \in \ker Si_*$ , and this follows from next commuting diagram with the Puppe exact sequence for  $C_{\rho}$  ([CS]):

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F^1(SD) & \xrightarrow{((S\rho_1)_*, (S\rho_2)_*)} & F^1(S\mathcal{B}_1 \oplus S\mathcal{B}_2) & \longrightarrow & F^1(C_{\rho}) \longrightarrow F^1(D) \\ & & \uparrow & & \uparrow (-S\rho_2)_*, (S\rho_1)_* & & (\iota_{\boxplus})_* \uparrow \\ & & \ker Si_* & \xrightarrow{(1, -1)} & F^1(SD \oplus SD) & \xrightarrow{(S\iota_{\theta})_*} & F^1(S\mathbb{K}(X_1 \boxplus X_2)). \end{array}$$

Here the commutativity of the left square follows from that  $(\rho_1 \oplus \rho_2): D \rightarrow \mathcal{B}_k$  factors through  $i_k: D \rightarrow A_k$  for each  $k = 1, 2$ .  $\square$

Similarly, one can show the exact sequence for  $KK^p(-, P)$ .

**Remark 6.14.** Let  $\text{ev}: C_i \rightarrow D$  be the evaluation map at 0 and  $\Delta: D \rightarrow D \oplus D$  be the diagonal embedding. In [FG1] Theorem C was shown by constructing an inverse element  $x \in KK(SA, C_i)$  of  $G$ . Thus, the connecting map  $KK(P, SA) \rightarrow KK(P, D)$  is given by  $x \otimes_{C_i} \text{ev}$ . In our proof, the semisplit extension

$$0 \longrightarrow \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) \longrightarrow C^*(\Theta(A)) + \mathbb{K}(\mathcal{F}(\mathfrak{X})J_{\mathfrak{X}}) \xrightarrow{s} A \longrightarrow 0.$$

gives the connecting map  $KK(P, A) \rightarrow KK(P, \mathcal{F}(\mathfrak{X})J_{\mathfrak{X}})$ . It might be interesting to compare two elements  $\text{ev} \otimes_D \Delta \otimes_{D \oplus D} \iota_\theta \in KK(C_i, \mathbb{K}(X_1 \boxplus X_2))$  and  $\delta_r \in KK^1(C_i, S\mathbb{K}(X_1 \boxplus X_2))$ .

We close the paper showing Theorem D.

*Proof of Theorem D.* When  $X_1^\circ$  and  $X_2^\circ$  are full, our proof above shows that the map  $(\phi \oplus \rho_1)_*$  from  $KK(\Delta\mathbf{T}(A, E), A \oplus D)$  to  $KK(\Delta\mathbf{T}(A, E), \Delta\mathbf{T}(A, E))$  is surjective. This implies the desired  $KK$ -equivalence by the following trick from [Pi1]: Take  $\gamma \in KK(\Delta\mathbf{T}(A, E), A \oplus D)$  such that  $1_{\Delta\mathbf{T}(A, E)} - (\alpha \oplus \delta) \otimes_{A \oplus D} (\phi \oplus \rho_1) = \gamma \otimes_{A \oplus D} (\phi \oplus \rho_1)$ . Since the left hand side is an idempotent in the ring  $KK(\Delta\mathbf{T}(A, E), \Delta\mathbf{T}(A, E))$ , it follows from Lemma 6.8 that  $\gamma \otimes_{A \oplus D} (\phi \oplus \rho_1) = 0$ . In the general case, one can check the surjectivity from the exact sequences in Theorem C for the reduced amalgamated free products  $(A, E)$  and  $(\Delta\mathbf{T}(A, E), \mathcal{E}) = (\mathcal{B}_1, \mathcal{E}_1) *_{D \oplus D} (\mathcal{B}_2, \mathcal{E}_2)$  (see §§ 4.4).  $\square$

## REFERENCES

- [Bo] B.H. Bowditch, Relatively hyperbolic groups. Preprint. 1999.
- [Br] L.G. Brown, Stable isomorphism of hereditary subalgebras of  $C^*$ -algebras. *Pacific J. Math.* **71** (1977), no. 2, 335–348.
- [CE] M.D. Choi and E.G. Effros, Nuclear  $C^*$ -algebras and injectivity: the general case, *Indiana Univ. Math. J.* **26** (1977), no. 3, 443–446.
- [CS] J. Cuntz and G. Skandalis, Mapping cones and exact sequences in  $KK$ -theory. *J. Operator Theory* **15** (1986), no. 1, 163–180.
- [Dy] K.J. Dykema, Exactness of reduced amalgamated free product  $C^*$ -algebras. *Forum Math.* **16** (2004), no. 2, 161–180.
- [DySh] K.J. Dykema and D. Shlyakhtenko, Exactness of Cuntz-Pimsner  $C^*$ -algebras. *Proc. Edinb. Math. Soc.* (2) **44** (2001), no. 2, 425–444.
- [DySm] K.J. Dykema and R.R. Smith, The completely bounded approximation property for extended Cuntz-Pimsner algebras. *Houston J. Math.* **31** (2005), no. 3, 829–840.
- [FF] P. Fima and A. Freslon, Graphs of quantum groups and  $K$ -amenability. *Adv. Math.* **260** (2014), 233–280.
- [FG1] P. Fima and E. Germain, The  $KK$ -theory of amalgamated free products, preprint, 2015.
- [FG2] P. Fima and E. Germain, The  $KK$ -theory of fundamental  $C^*$ -algebras, preprint, 2015.
- [Ha] K. Hasegawa,  $KK$ -equivalence for Amalgamated Free Product  $C^*$ -algebras, *Int. Math. Res. Not. IMRN*, to appear.
- [JV] P. Julg and A. Valette,  $K$ -theoretic amenability for  $\text{SL}_2(\mathbb{Q}_p)$ , and the action on the associated tree, *J. Funct. Anal.* **58** (2) (1984) 194–215.
- [Ka] T. Katsura, On  $C^*$ -algebras associated with  $C^*$ -correspondences. *J. Funct. Anal.* **217** (2004), no. 2, 366–401.
- [Ki1] E. Kirchberg, Commutants of unitaries in UHF algebras and functorial properties of exactness. *J. Reine Angew. Math.* **452** (1994), 39–77.
- [Ki2] E. Kirchberg, On subalgebras of the CAR-algebra. *J. Funct. Anal.* **129** (1995), no. 1, 35–63.
- [Ku] A. Kumjian, On certain Cuntz-Pimsner algebras. *Pacific J. Math.* **217** (2004), no. 2, 275–289.
- [La] E.C. Lance, *Hilbert  $C^*$ -modules. A toolkit for operator algebraists*. London Mathematical Society Lecture Note Series, **210**. Cambridge University Press, Cambridge, 1995.
- [Ok] R. Okayasu, Cuntz-Krieger-Pimsner algebras associated with amalgamated free product groups. *Publ. Res. Inst. Math. Sci.* **38** (2002), no. 1, 147–190.
- [Oz1] N. Ozawa, Nuclearity of reduced amalgamated free product  $C^*$ -algebras. *Sūrikaiseikikenkyūsho Kōkyūroku* No. **1250** (2002), 49–55.
- [Oz2] N. Ozawa, Boundary amenability of relatively hyperbolic groups. *Topology Appl.* **153** (2006), no. 14, 2624–2630.
- [Pi1] M.V. Pimsner,  $KK$ -groups of crossed products by groups acting on trees. *Invent. Math.* **86** (1986), no. 3, 603–634.
- [Pi2] M.V. Pimsner, A class of  $C^*$ -algebras generalizing both Cuntz-Krieger algebras and crossed products by  $\mathbb{Z}$ . in *Free probability theory*, 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997.

- [PV] M.V. Pimsner and D.V. Voiculescu,  $K$ -groups of reduced crossed products by free groups. *J. Operator Theory* **8** (1982), no. 1, 131–156.
- [Se] J.P. Serre, *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
- [Sk] G. Skandalis, Exact sequences for the Kasparov groups of graded algebras. *Canad. J. Math.* **37** (1985), no. 2, 193–216.
- [Sk2] G. Skandalis, Une notion de nucléarité en  $K$ -théorie (d’après J. Cuntz). *K-Theory* **1**, no. 6 (1988): 549–573.
- [Spi] J. Spielberg, Free-product groups, Cuntz–Krieger algebras, and covariant maps. *Internat. J. Math.* **2** (1991), no. 4, 457–476.
- [Va] A. Valette, *Introduction to the Baum-Connes conjecture*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002.
- [Ve] R. Vergnioux,  $K$ -amenability for amalgamated free products of amenable discrete quantum groups. *J. Funct. Anal.* **212** (2004), no. 1, 206–221.
- [Vo] D. Voiculescu, Symmetries of some reduced free product  $C^*$ -algebras. in *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Math. 1132, Springer, Berlin, (1985): 556–588.

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